Topics in Multi-Dimensional Calculus of Variations

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Scientific abstract

In this report we week to establish conditions for a certain parameterised family of polyconvex functionals to have a global minimum. These functionals arise from considering the Dirichlet energy functional and relate to the study of elasticity. The functionals we will be considering do not meet the usual conditions to be treated by the direct method in the calculus of variations and so we will make use of other techniques including partial differential inclusions and Fourier-type analysis. We will develop several techniques for solving PDIs and evaluate the benefits of these methods in relation to analysing the functionals under consideration. Overall, we will obtain bounds on a parameter in both the cases of sufficient and necessary conditions for the non-negativity of the functional, which is equivalent to showing that zero is the global minimum.

Keywords and AMS Classification Codes: calculus of variatons; Dirichlet energy; polyconvexity; partial differential inclusions; problem of potential wells

Notation

All notation in this report is understood to be standard, along with the following:

- We shall denote the two dimensional unit ball in the standard Euclidean norm by B and the square $[-2,+2]^2$ by Q. An arbitrary bounded domain in $\mathbb{R}^d$ will be denoted by Ω.
- The orthogonal and special orthogonal groups are

$$\begin{aligned} \mathrm{O}(d) &= \{M \in \mathbb{R}^{d\times d} : M^\mathsf{T} M = I_d\}, \\ \mathrm{SO}(d) &= \{M \in \mathbb{R}^{d\times d} : M^\mathsf{T} M = I_d, \det M = 1\}. \end{aligned}$$

- All vector and matrix norms will be denoted by $|\cdot|_*$ with the $*$ being $p \geqslant 1$ for the operator norm induced by the ℓ^p norm (or just the ℓ^p norm itself in the case of a vector norm) or F for the Frobenius norm. Similarly, function space norms will be denoted using $\|\cdot\|_*$.
- We will write mean integrals as

$$\fint_\Omega f(x)\,\mathrm{d}x := \frac{1}{|\Omega|}\int_\Omega f(x)\,\mathrm{d}x$$

- The matrices $\mathrm{diag}(\pm 1, 1)$ shall be denoted by $I_{\pm 1}$ and we let

$$J = \begin{pmatrix} 0 & -1 \\ +1 & 0 \end{pmatrix}$$

  We shall define $\theta(a) \in [0, 2\pi)$ to be the counter-clockwise angle the vector $a \in \mathbb{R}^2$ makes with the positive x-axis. For $\theta \in \mathbb{R}$, let $R(\theta)$ denote the standard matrix for rotating a vector in $\mathbb{R}^2$ counter-clockwise by θ radians about the origin.

$$R(\theta) = \begin{pmatrix} +\cos(\theta) & -\sin(\theta) \\ +\sin(\theta) & +\cos(\theta) \end{pmatrix}$$

  For a vector $a \in \mathbb{R}^2$, we then define

$$R(a) := R(\theta(a)) = \frac{1}{|a|^2}\begin{pmatrix} a & Ja \end{pmatrix}$$

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1 Introduction

1.1. Motivation & Literature Review

We shall start by considering the Dirichlet energy functional $F : \mathcal{A} \to \mathbb{R}$, given by

$$F(u) = \int_B |\nabla u|_F^2 \, dx$$

with the space of admissible functions being

$$\mathcal{A} = \{u \in u_0 + W_0^{1,2}(B; \mathbb{R}^2) : \det \nabla u = 1 \text{ a.e. } \}$$

where u_0 is the so-called double-covering map. It can be shown that u_0 is in fact a stationary point of F [6]. In analysing F, it is beneficial to introduce an excess functional E that is defined by

$$F(u_0 + u) = F(u_0) + E(u)$$

If F were linear, the existence of E would be trivial as we would simply have that $E = F$. However, for nonlinear F, such as the Dirichlet energy functional, the existence of an excess functional is not guaranteed. Here, we find that

$$E(u) = \int_B |\nabla u|_F^2 + f(x) \det \nabla u \, dx =: E_f(u)$$

with $f(x) = 3 \log |x|_2$ [6]. When we have a functional of this form, we often refer to f as the pressure function, which acts as a Lagrange multiplier for the constraint $\det \nabla u = 1$ (incompressibility [4]). We observe that E_f takes the form

$$E_f(u) = \int_\Omega W(x, \nabla u) \, dx \qquad W : \Omega \times \mathbb{R}^{2\times 2} \to \mathbb{R}$$

with W being polyconvex [3, 4, 10]. In other words $W(x_0, A) = \phi_{x_0}(A, \det A)$ with $\phi_{x_0} : \mathbb{R}^{2\times 2} \times \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ is convex, for each $x_0 \in \Omega$. The minimisation of such functionals is often treated by the direct method in the calculus of variations [2,10] and a typical hypothesis is some form of coercivity condition [9,14,15], a lower bound that encodes the growth rate of W in A, to show the existence of a global minimiser. For example,

$$\alpha_1 |A|_F^2 + \gamma_1(x) \leqslant W(x, A) \leqslant \alpha_2 |A|_F^2 + \gamma_2(x) \qquad \alpha_2 \geqslant \alpha_1 > 0 \qquad \gamma_1, \gamma_2 \in L^1(\Omega; \mathbb{R})$$

is sufficient for E_f to have a global minimiser in $W_0^{1,2}(\Omega; \mathbb{R}^2)$ [10]. However, in our case of

$$W(x, A) = |A|_F^2 + f(x) \det A$$

we may not even have that W is bounded below (consider f with $x_* \in \Omega$ such that $f(x_*) \leqslant -4$). This motivates the exploration of novel techniques to minimise functionals of the form E_f. In particular, we shall show that 0 is the global minimum of the functional

$$E_f(u) = \int_B |\nabla u|_F^2 + f(x) \det \nabla u \, dx \qquad u \in W_0^{1,2}(B; \mathbb{R}^2)$$

with $f(x) = \lambda |x|_2$ for particular values of $\lambda \in \mathbb{R}^+$. Since constant u satisfy $E_f(u) = 0$, it is sufficient to show that $E_f \geqslant 0$. This is due to the fact that just a single u with $E_f(u) < 0$ would imply that $E_f(ku) = k^2 E_f(u) \to -\infty$ as $k \to \infty$, making E_f unbounded below. We will now investigate how the choice of the parameter λ affects the bounding of E_f.

1.2. Sufficient Conditions for Non-Negativity

We shall first establish sufficient conditions on λ for non-negativity of the functional E_f. We can do this quite crudely by making use of Hadamard's inequality (see 1.3):

$$\begin{aligned} E_f(u) &= \int_B |\nabla u(x)|_F^2 + \lambda |x|_2 \det \nabla u(x) \, dx \\ &\geqslant \int_B |\nabla u(x)|_F^2 - \frac{\lambda}{2} |x|_2 |\nabla u(x)|_F^2 \, dx \\ &= \int_B \left(1 - \frac{\lambda}{2} |x|_2\right) |\nabla u(x)|_F^2 \, dx \geqslant 0 \end{aligned}$$

for $\lambda \leqslant 2$. Hence $\lambda \leqslant 2$ is sufficient for non-negativity. We shall now seek to improve upon this bound by considering a Fourier-like decomposition of the variable u, along with a weighted Poincaré inequality.

Lemma 1.1. *For $f \in W^{1,2}((0,1); \mathbb{R}^2)$ satisfying $f(1) = 0$, we have*

$$\int_0^1 r |f(r)|_2^2 \, dr \leqslant \frac{1}{j_0^2} \int_0^1 r \left|f'(r)\right|_2^2 \, dr$$

where j_0 is the first zero of the Bessel function J_0.

Proof. We shall start by considering f satisfying $\int_0^1 r\,|f(r)|_2^2 \,\mathrm{d}r = 1$. Then we shall minimize the functional

$$f \mapsto \int_0^1 r\left|f'(r)\right|_2^2 \,\mathrm{d}r \qquad f(1) = 0$$

subject to $\int_0^1 r\,|f(r)|_2^2 \,\mathrm{d}r = 1$. To this end, we define the functional

$$\Lambda[f] = \int_0^1 rf'(r)^2 \,\mathrm{d}r + \mu\left(\int_0^1 rf(r)^2 \,\mathrm{d}r - 1\right) \qquad f(1) = 0$$

with a parameter $\mu \in \mathbb{R}$. Here we have assumed f is scalar-valued as this is sufficient. We then calculate

$$\begin{aligned}\Lambda[f + \epsilon\varphi] &= \int_0^1 r(f' + \epsilon\varphi')^2 \,\mathrm{d}r + \mu\left(\int_0^1 r(f + \epsilon\varphi)^2 \,\mathrm{d}r - 1\right) \\ &= \Lambda[f] + 2\epsilon\int_0^1 rf'\varphi' + \mu rf\varphi \,\mathrm{d}r + \epsilon^2\int_0^1 r(\varphi')^2 + \mu r\varphi^2 \,\mathrm{d}r\end{aligned}$$

for arbitrary $\varphi \in C^\infty((0,1);\mathbb{R})$ satisfying $\varphi(1) = 0$. From this we can find a stationary function f by making the first variation zero, i.e:

$$\begin{aligned}0 &= \int_0^1 rf'\varphi' + \mu rf\varphi \,\mathrm{d}r \\ &= \int_0^1 (rf'\varphi)' - (rf')'\varphi + \mu rf\varphi \,\mathrm{d}r \\ &= \int_0^1 (\mu rf - (rf')')\varphi \,\mathrm{d}r \qquad \forall\varphi,\end{aligned}$$

where we assume that

$$\lim_{r\to 0^+} rf'(r) = 0.$$

If this doesn't hold, then $rf'(r) \sim A \neq 0$ as $r \to 0$ and so $\int_0^1 rf'(r)^2 \,\mathrm{d}r \sim \int_0^1 \frac{A^2}{r} \,\mathrm{d}r = +\infty$.Now we must solve

$$\mu rf - (rf')' = 0,$$

or equivalently

$$r^2 f'' + rf' - \mu r^2 f = 0.$$

This can be solved using Bessel functions to yield

$$f(r) = \alpha J_0(+i\sqrt{\mu}r) + \beta Y_0(-i\sqrt{\mu}r)$$

for appropriate constants α, β. We know that

$$\begin{aligned}\lim_{r\to 0^+} rJ_0'(r) &= 0, \\ \lim_{r\to 0^+} rY_0'(r) &\neq 0.\end{aligned}$$

Hence we take $\beta = 0$. To satisfy $f(1) = 0$, we need

$$\alpha J_0(i\sqrt{\mu}) = 0.$$

Since we do not wish to take $\alpha = 0$ ($f = 0$ will not satisfy the constraint), we must have that $\mu = -j_0^2$ for j_0 a zero of J_0. Finally we plug $f(r) = \alpha J_0(j_0 r)$ into the constraint:

$$\begin{aligned} 1 &= \int_0^1 r f(r)^2 \, \mathrm{d}r \\ &= \alpha^2 \int_0^1 r J_0(j_0 r)^2 \, \mathrm{d}r \\ &= \frac{\alpha^2}{j_0^2} \int_0^{j_0} r J_0(r)^2 \, \mathrm{d}r \\ &= \frac{\alpha^2}{j_0^2} \left[\frac{r^2}{2} \left(J_0(r)^2 + J_1(r)^2 \right) \right]_0^{j_0} \\ &= \frac{\alpha^2}{j_0^2} \frac{j_0^2}{2} J_1(j_0)^2 = \frac{(\alpha J_1(j_0))^2}{2}. \end{aligned}$$

Hence $\alpha = \pm \frac{\sqrt{2}}{J_1(j_0)}$ and so

$$f(r) = \pm\sqrt{2} \frac{J_0(j_0 r)}{J_1(j_0)} \qquad \mu = -j_0^2.$$

Then

$$\begin{aligned} \Lambda[f] &= \int_0^1 r f'(r)^2 \, \mathrm{d}r \\ &= \frac{2 j_0^2}{J_1(j_0)^2} \int_0^1 r J_0'(j_0 r)^2 \, \mathrm{d}r \\ &= \frac{2}{J_1(j_0)^2} \int_0^{j_0} r J_0'(r)^2 \, \mathrm{d}r \\ &= \frac{2}{J_1(j_0)^2} \int_0^{j_0} r J_1(r)^2 \, \mathrm{d}r \\ &= \frac{2}{J_1(j_0)^2} \left[\frac{r^2}{2} \left(J_1(r)^2 - J_0(r) J_2(r) \right) \right]_0^{j_0} \\ &= \frac{2}{J_1(j_0)^2} \frac{j_0^2}{2} J_1(j_0)^2 = j_0^2. \end{aligned}$$

Since we wish to minimise this quantity, we shall in fact take j_0 to be the first zero of J_0. Thus we have

$$\int_0^1 r f'(r)^2 \, \mathrm{d}r \geqslant j_0^2 \qquad \int_0^1 r f(r)^2 \, \mathrm{d}r = 1.$$

Now consider arbitrary $f \in W^{1,2}(0,1);\mathbb{R})$ with $f(1) = 0$. Then define $\tilde{f} \in W^{1,2}((0,1);\mathbb{R})$ by

$$\tilde{f}(r) = \frac{f(r)}{\sqrt{\int_0^1 s f(s)^2 \, \mathrm{d}s}}.$$

Then $\tilde{f}(1) = 0$ and $\int_0^1 rf(r)^2\,\mathrm{d}r = 1$. Hence we have that

$$\int_0^1 r\tilde{f}'(r)^2\,\mathrm{d}r = \frac{\int_0^1 rf'(r)^2\,\mathrm{d}r}{\int_0^1 rf(r)^2\,\mathrm{d}r} \geqslant j_0^2,$$

which gives the desired inequality

$$\int_0^1 rf(r)^2\,\mathrm{d}r \leqslant \frac{1}{j_0^2}\int_0^1 rf'(r)^2\,\mathrm{d}r.$$

□

Now we perform a Fourier-like decomposition of u as

$$u = \sum_{j\geqslant 0} u^{(j)} = \frac{1}{2}A_0(r) + \sum_{j>0} A_j(r)\cos(j\theta) + B_j(r)\sin(j\theta)$$

with the $A_j, B_j : [0,1] \to \mathbb{R}^2$ satisfying $A_j(1) = B_j(1) = 0$ to ensure $u|_{\partial B} = 0$. We then use the orthogonality of the $\cos(j\theta)$ and $\sin(j\theta)$ in $L^2((0,2\pi);\mathbb{R})$ to get the weighted Poincaré inequality

$$\begin{aligned}
\int_B \left|u^{(j)}\right|_2^2\,\mathrm{d}x &= \int_0^1 \pi\left(\left|A_j(r)\right|_2^2 + \left|B_j(r)\right|_2^2\right) r\,\mathrm{d}r \\
&\leqslant \frac{1}{j_0^2}\int_0^1 \pi\left(\left|A_j'(r)\right|_2^2 + \left|B_j'(r)\right|_2^2\right) r\,\mathrm{d}r \\
&= \frac{1}{j_0^2}\int_B \left|u_{,r}^{(j)}\right|_2^2\,\mathrm{d}x \qquad j \geqslant 1
\end{aligned}$$

for Fourier modes. Similar calculations show that the inequality also holds for $j = 0$. We can use this, along with Lemma 3.2 in [6], to deduce that

$$\begin{aligned}
E_f(u) &= \int_B |\nabla u|_F^2 + f(x)\det\nabla u(x)\,\mathrm{d}x \\
&= \sum_{j\geqslant 0}\int_B \left|u_{,r}^{(j)}\right|_2^2 + \left|u_{,\tau}^{(j)}\right|_2^2 + \frac{\lambda}{2}f_0'(|x|_2)\left\langle u^{(j)}, Ju_{,\tau}^{(j)}\right\rangle_2\,\mathrm{d}x \\
&\geqslant \sum_{j\geqslant 0}\int_B j_0^2\left|u^{(j)}\right|_2^2 + \left|u_{,\tau}^{(j)}\right|_2^2 - \frac{\lambda}{2}\left|u^{(j)}\right|_2\left|u_{,\tau}^{(j)}\right|_2\,\mathrm{d}x \\
&= \sum_{j\geqslant 0}\int_B \begin{pmatrix}\left|u^{(j)}\right|_2 \\ \left|u_{,\tau}^{(j)}\right|_2\end{pmatrix}^{\mathsf{T}} M \begin{pmatrix}\left|u^{(j)}\right|_2 \\ \left|u_{,\tau}^{(j)}\right|_2\end{pmatrix}\mathrm{d}x \qquad M = \begin{pmatrix} j_0^2 & -\frac{\lambda}{4} \\ -\frac{\lambda}{4} & 1\end{pmatrix}.
\end{aligned}$$

Then it is sufficient to show that M is positive-definite for non-negativity. Since M is symmetric with $\operatorname{tr} M = 1 + j_0^2 > 0$, it suffices to check $\det M \geqslant 0$. Here we have $\det M = j_0^2 - \left(\frac{\lambda}{4}\right)^2$ so $\lambda \leqslant 4j_0 \approx 9.619$ is sufficient. This gives us a true improvement of the bound $\lambda \leqslant 2$ that is obtained by just using the pointwise Hadamard's inequality. This method could most likely be adapted for different pressure functionals f, provided they are radially symmetric.

1.3. Necessary Conditions for Non-Negativity

We shall now seek a necessary condition on λ by picking a $u \in W_0^{1,2}(B;\mathbb{R}^2)$ and varying λ, observing when $E_f(u)$ switches sign. We shall do this by considering a u that is zero on a subset $K \subset B$ and satisfies $\nabla u \in \mathrm{O}(2)$ almost everywhere in a countable disjoint collection of squares that all sit inside B. For simplicity, we shall just consider squares that are aligned with the axes (i.e: not rotated). On each of these squares, we will then take u to be an appropriately shifted and scaled copy of a function $u^* : Q \to \mathbb{R}^2$ that satisfies $\nabla u^* \in \mathrm{O}(2)$ almost everywhere in Q (we shall defer the discussion of the existence of such a u^* for now). If we label each of these squares as Q_i, with width w_i and centre c_i. Then we have that [5]

$$
\begin{aligned}
E_f(u) &= \sum_i \int_{Q_i} 2 + \mathbf{1}_{Q_i^+}(y) f(y) - \mathbf{1}_{Q_i^-}(y) f(y)\,\mathrm{d}y \\
&= \sum_i \left(2|Q_i| + \int_{Q_i^+} f(y)\,\mathrm{d}y - \int_{Q_i^-} f(y)\,\mathrm{d}y \right) \\
&= \sum_i \left(2|Q_i| + \int_{Q^+} f\left(\frac{w_i}{4}x + c_i\right)\mathrm{d}x - \int_{Q^-} f\left(\frac{w_i}{4}x + c_i\right)\mathrm{d}x \right) \\
&= 2(|B| - |K|) + \int_{Q^+} \sum_i f\left(\frac{w_i}{4}x + c_i\right)\mathrm{d}x - \int_{Q^-} \sum_i f\left(\frac{w_i}{4}x + c_i\right)\mathrm{d}x \\
&= 2(\pi - |K|) + \lambda \left(\int_{Q^+} F_0(x)\,\mathrm{d}x - \int_{Q^-} F_0(x)\,\mathrm{d}x \right),
\end{aligned}
$$

where $F_0 : Q \to \mathbb{R}^2$ is given by $F_0(x) = \sum_i f_0\left(\frac{w_i}{4}x + c_i\right)$ and we have defined

$$
Q^{\pm} = \{x \in Q : \det \nabla u^*(x) = \pm 1\}
$$

and similarly for $Q_i^{\pm}$. Thus $E_f(u) \geqslant 0$ requires

$$
+\lambda I_0 \geqslant -2(\pi - |K|),
$$

where $I_0 = \int_{Q^+} F_0(x)\,\mathrm{d}x - \int_{Q^-} F_0(x)\,\mathrm{d}x$. We could repeat this calculation with $u^* \circ I_{-1}$ instead of $u_* \circ I_{+1} = u^*$. We would then require

$$
-\lambda I_0 \geqslant -2(\pi - |K|).
$$

Thus, for non-negativity, we require

$$
\lambda \leqslant \frac{2(\pi - |K|)}{|I_0|}.
$$

Here $|I_0|$ and $|K|$ are parameters that depend on how we tile the ball B with squares Q_i and also the choice of u^*. We can, in fact take $|K| \to \pi$, although this would require u to be identically zero. Then $E_f(u) = 0$ regardless of the choice of λ. We can also take $|K| \to 0$. This requires B to be tiled by squares in its entirety. This is possible using a countably infinite number of squares but it will also make calculating $|I_0|$ more difficult. To calculate $|I_0|$, we will now have to investigate the existence of u^* and explain how to construct it.

1.4. Partial Differential Inclusions

We shall now clarify the existence of a solution $u \in W^{1,\infty}(\Omega;\mathbb{R}^d) \subset W^{1,2}(\Omega;\mathbb{R}^d)$ to

$$\begin{aligned} \nabla u(x) &\in \mathrm{O}(d) \qquad \text{a.e. } x \in \Omega \\ u(x) &= 0 \qquad\qquad \forall x \in \partial\Omega. \end{aligned} \tag{1.1}$$

Note that this is a PDI (partial differential inclusion), rather than a PDE. Such inclusions have been studied extensively in the context of phase transformations [1, 16–18]. This PDI is a particular example of the so called **problem of potential wells** [8, 18, 19], namely finding a $u \in \varphi + W_0^{1,\infty}(\Omega;\mathbb{R}^d)$ such that

$$\nabla u(x) \in \bigcup_{i=1}^{N} \mathrm{SO}(d)A_i \qquad \text{a.e. } x \in \Omega$$

for fixed $A_i \in \mathbb{R}^{d\times d}$. In our case we take just two wells, given by $A_1 = I_{+1}$ and $A_2 = I_{-1}$, with $\varphi \equiv 0$. We will mainly be interested in the case of $d = 2$, in which we can characterise the two wells using the following result from Dacorogna and Marcellini [11].

Lemma 1.2. *([11, Proposition 7.6]) Let $M \in \mathbb{R}^{2\times 2}$. Then the singular values of M are given by*

$$\begin{aligned} \sigma_1(M) &= \frac{1}{2}\sqrt{|M|_F^2 + 2\,|\det M|} - \frac{1}{2}\sqrt{|M|_F^2 - 2\,|\det M|}, \\ \sigma_2(M) &= \frac{1}{2}\sqrt{|M|_F^2 + 2\,|\det M|} + \frac{1}{2}\sqrt{|M|_F^2 - 2\,|\det M|}. \end{aligned}$$

Since we know that the singular values of real matrices are real, we get an important corollary.

Theorem 1.3 (Hadamard's inequality)**.** *Let $M \in \mathbb{R}^{2\times 2}$. Then $|M|_F^2 \geqslant 2\,|\det M|$.*

We can then use the lemma to characterise the potential wells using just the determinant and Frobenius norm.

Proposition 1.4. *We have the following equivalent definition of* $\mathrm{SO}(2)I_{\pm 1}$

$$\mathrm{SO}(2)I^{\pm} = \{M \in \mathbb{R}^{2\times 2} : |M|_F = \sqrt{2},\ \det(M) = \pm 1\}.$$

Proof. We have that $M \in \mathrm{O}(2)$ iff $M^{\mathsf{T}}M = I_{+1}$. This holds iff the eigenvalues of $M^{\mathsf{T}}M$ are both 1, i.e: $\sigma_1(M) = \sigma_2(M) = 1$. If we add/subtract these two conditions using the above proposition, we obtain

$$\begin{aligned} \sqrt{|M|_F^2 + 2\,|\det M|} &= 2, \\ \sqrt{|M|_F^2 - 2\,|\det M|} &= 0. \end{aligned}$$

Hence

$$|M|_F^2 = 2\,|\det M|\,, \qquad |\det M| = 1.$$

Hence we have that $M \in \mathrm{O}(2)$ iff $|M|_F = \sqrt{2}$ and $\det M = \pm 1$. □

1.5. Existence of Solutions

The existence of solutions to the problem of potential wells (as well as generalisations) has been investigated extensively by Dacorogna and Marcellini [11]. In particular, we get the following existence theorem.

Theorem 1.5. *([11, Theorem 7.28]) Let $\varphi \in W^{1,\infty}(\Omega;\mathbb{R}^d)$ be such that* $\operatorname{ess\,sup}|\nabla\varphi|_2 < 1$. *Then there exists a dense (in the L^∞ norm) set of solutions $u \in W^{1,\infty}(\Omega;\mathbb{R}^d)$ to*

$$\begin{aligned} \nabla u(x) &\in \mathrm{O}(d) \qquad \textit{a.e. } x \in \Omega, \\ u(x) &= \varphi(x) \qquad \forall x \in \partial\Omega. \end{aligned} \tag{1.2}$$

We then observe the following:

- Since $\det \nabla(\cdot)$ is a null Lagrangian [11, 8.47] [9], we have

$$|\Omega^+| - |\Omega^-| = \int_\Omega \det \nabla u \,\mathrm{d}x = \int_\Omega \det \nabla\varphi \,\mathrm{d}x.$$

- Since the $\Omega^\pm$ form a partition of Ω (up to a set of zero measure), we have

$$|\Omega^+| + |\Omega^-| = |\Omega|.$$

Hence we have deduced that

$$|\Omega^\pm| = \frac{|\Omega|}{2}\left(1 \pm \fint_\Omega \det\nabla\varphi\,\mathrm{d}x\right)$$

and so a necessary condition on φ is

$$\left|\fint_\Omega \det\nabla\varphi\,\mathrm{d}x\right| < 1.$$

We can show that this condition is satisfied by any φ that satisfies the assumptions in Theorem 1.5 by using Lemma 1.2 along with Hadamard's inequality:

$$\begin{aligned} \sqrt{|\det\nabla\varphi|} &= \frac{1}{2}\sqrt{2\,|\det\nabla\varphi| + 2\,|\det\nabla\varphi|} \quad + \frac{1}{2}\sqrt{2\,|\det\nabla\varphi| - 2\,|\det\nabla\varphi|} \\ &\leqslant \frac{1}{2}\sqrt{|\nabla\varphi|_F^2 + 2\,|\det\nabla\varphi|} \qquad + \frac{1}{2}\sqrt{|\nabla\varphi|_F^2 - 2\,|\det\nabla\varphi|} \qquad = |\nabla\varphi|_2 < 1. \end{aligned}$$

Hence $|\det\nabla\varphi| < 1$ a.e. and so $\left|\fint_\Omega \det\nabla\varphi\,\mathrm{d}x\right| \leqslant \fint_\Omega |\det\nabla\varphi|\;\mathrm{d}x < 1$.

Remark 1.6. *Note that φ constant will always satisfy the necessary conditions and give an equal distribution of gradients between the two wells, i.e: $|\Omega^{\pm}| = \frac{|\Omega|}{2}$.*

Now, in the case of $d = 2$, consider a solution $u \in \varphi + W_0^{2,2}(\Omega; \mathbb{R}^d)$ (we have now assumed the existence of an extra derivative) to 1.2. The PDI can be written as an equivalent system of PDEs

$$\nabla u_i \cdot \nabla u_j = \delta_{ij} \qquad \forall i, j = 1, \ldots, d.$$

Now Hölder's inequality allows us to use the following product rule:

$$0 = \nabla(\nabla u_i \cdot \nabla u_i) = 2\nabla u_i \cdot \nabla^{\otimes 2} u_i.$$

Since $\nabla u_i \neq 0$, we have that $\det \nabla^{\otimes 2} u_i = 0$. Now consider $i \neq j$ so

$$\nabla u_i \cdot \nabla u_j = 0 \Rightarrow \nabla u_i = \pm \nabla u_j \times \hat{z} = \pm \nabla \times u_j \hat{z}.$$

so $\operatorname{tr} \nabla^{\otimes 2} u_i = \nabla^2 u_i = 0$. Hence we must have $\nabla^{\otimes 2} u_i = 0$ and so u is piecewise affine. Fortunately, it turns out neither the restriction to the case $d = 2$ nor the assumption of a second derivative are required to show u is piecewise affine. This result is summarised in **Liouville's theorem**.

Theorem 1.7. *([18, Theorem 2.4]) Suppose that $\nabla u \in SO(d)$ a.e. in Ω. Then ∇u is equal a.e. to a constant matrix and we can write $u(x) = Mx + c$ for $M \in SO(d)$ and $c \in \mathbb{R}^d$.*

This immediately generalises to the case of $SO(d)I_{\pm 1}$. We then get the following corollary.

Corollary 1.8. *Suppose that $\nabla u \in O(d)$ a.e. in Ω. Then ∇u is piecewise constant.*

Thus, when constructing solutions to 1.2, it is sufficient to consider only piecewise affine solutions.

1.6. Solutions in One Dimension

In one dimension, an open subset $\Omega \subset \mathbb{R}$ is just a union of open intervals and so, without loss of generality, we will set $\Omega = (a, b)$ for $a < b$. The differential inclusion with a general boundary condition is then given by

$$u'(x) \in \{-1, +1\} \text{ a.e. } x \in (a, b), \qquad u(a) = A,\ u(b) = B.$$

This is an example of the so-called **Eikonal equation** [12]. The constraint on the boundary data can be expressed compactly as:

$$|B - A| < |b - a|.$$

Here the geometric need for this constraint is more apparent. In the limiting case of $|B-A| = |b-a|$, we obtain a unique solution that is affine. Otherwise, we must instead consider a piecewise affine solution.

1.6.1. Two-Piece Solutions

Considering that we cannot have an affine solution, the next simplest type of solution to look for is a piecewise affine solution with a single peak or trough. Consider the following:

$$u_{\pm}(x) = \begin{cases} A \pm (x-a) & x \in [a, \overline{x}] \\ B \mp (x-b) & x \in [\overline{x}, b] \end{cases} \qquad \overline{x} := \frac{(a+b) \pm (B-A)}{2}.$$

Then u_+ is a solution with a single peak and u_- is a solution with a single trough. In the limiting case of $|B-A| = |b-a|$, we recover the affine solution mentioned previously.

1.6.2. Multi-Piece Solutions

We now generalise to $n \in \mathbb{N}$ pieces. Consider a partition $(x_0, \ldots, x_n)$ of $[a, b]$ and a tuple $\alpha \in \{-1, +1\}^n$. We construct a candidate solution u_α given by

$$u_\alpha(x) = A_{i-1} + \alpha_i(x - x_{i-1}) \quad x \in [x_{i-1}, x_i] \quad i = 1, \ldots, n \tag{1.3}$$

with $A_0 = A$ and $A_n = B$. Then, by continuity, we obtain the following recurrence relation for the A_i:

$$A_i - A_{i-1} = \alpha_i(x_i - x_{i-1}) \qquad i = 1, \ldots, n. \tag{1.4}$$

Then we can sum both sides form $i = 1$ to $i = n$ to find that

$$B - A = \sum_{i=1}^{n} \alpha_i(x_i - x_{i-1}) = \sum_{i=1}^{n} \alpha_i \Delta x_i \tag{1.5}$$

where $\Delta x_i = x_i - x_{i-1}$. Then, for any configuration α, we can pick a partition $(x_0, \ldots, x_n)$ of $[a, b]$ satisfying 1.5 and hence obtain the A_i through 1.4 to give a solution u_α via 1.3.

1.6.3. Application using Polar Coordinates

We can consider another example of an Eikonal equation, namely

$$|\nabla u(x)|_2 = 1 \text{ a.e. } x \in B \qquad u|_{\partial B} = u_0 \in \mathbb{R}.$$

We can find a solution $u(r, \theta) = u(r)$, given in terms of polar coordinates, by solving

$$|u'(r)| = 1 \text{ a.e. } r \in (0, 1) \qquad u(0) = 0,\ u(1) = u_0$$

using the methods from the previous sections. However, this only works when the boundary data is constant. If we try to generalise this to a variable boundary condition, different techniques will be required.

2
Solutions in Two Dimensions

2.1. Explicit Construction of a Particular Solution to the PDI

We shall now set out to construct an explicit solution 1.1 in the case of $d = 2$ and $\Omega = Q$. This construction could then be generalised to other bounded domains in $\mathbb{R}^2$ (for example, one can use Vitali covering results [13] or the Riemann mapping theorem [7]). This construction will use a similar approach to the explicit solution given by Cellina and Perrotta [8] for the case of $d = 3$.
Given an $x = (x_1, x_2) \in \mathbb{R}^2$, we define the functions $|X_i|(x), |X_s|(x)$ to be the smallest and largest values in $(|x_1|, |x_2|)$ respectively. These are continuous functions that are invariant under permutation and taking absolute values of the entries of x.
Next, we define the functions $i, s : \mathbb{R}^2 \to \{1, 2\}$ to give the position of the smallest and largest values in $(|x_1|, |x_2|)$ respectively. These functions are locally constant.
Now define $f^k : \mathbb{R} \to \mathbb{R}$ (for $k \in \mathbb{Z}^*$) by taking f^1 to be the 1-periodic extension of

$$t \mapsto \inf\{t, 1-t\}, \quad t \in [0,1]$$

and

$$f^k(t) = \frac{1}{2^{k-1}} f^1\left(2^{k-1} t\right).$$

Note that these are all even continuous functions. We then define a function $u^1 : \frac{1}{2}\overline{Q} \to \mathbb{R}^2$ by

$$u_1^1(x) = \inf\{f^1(|X_i|(x)), f^1(|X_s|(x))\},$$
$$u_2^1(x) = \begin{cases} f^2(|X_s|(x)) & |X_i|(x) + |X_s|(x) \leqslant 1, \\ f^2(|X_i|(x)) & |X_i|(x) + |X_s|(x) \geqslant 1. \end{cases}$$

One can verify that u^1 is well-defined and Lipschitz on $\frac{1}{2}Q$.

Lemma 2.1. *([8, Claim 1]) For all $x_1 \in (-1, +1)$, we have $u^1(x_1, 1) = u^1(x_1 \bmod 1, 1)$.*

Proof. Taking $x = (x_1, 1)$ with $x_1 \in (-1, +1)$, we have

$$\begin{aligned} |X_i|(x) &= |x_1|, \\ |X_s|(x) &= 1, \end{aligned}$$

Thus we obtain

$$\begin{aligned} u_1^1(x_1, 1) &= \inf\{f^1(|x_1|), f^1(1)\}, \\ u_2^1(x_1, 1) &= \begin{cases} f^2(1) & |x_1| = 0, \\ f^2(|x_1|) & |x_1| > 0. \end{cases} \end{aligned}$$

Now we use the that f^1, f^2 are non-negative, even and satisfy

$$f^1(1) = f^2(1) = 0.$$

Hence

$$\begin{aligned} u_1^1(x_1, 1) &= \inf\{f^1(|x_1|), 0\}, \\ u_2^1(x_1, 1) &= \begin{cases} 0 & |x_1| = 0, \\ f^2(|x_1|) & |x_1| > 0. \end{cases} \end{aligned}$$

Finally, we have

$$\begin{aligned} u_1^1(x_1, 1) &= 0, \\ u_2^1(x_1, 1) &= f^2(x_1). \end{aligned}$$

Clearly $u^1(x_1, 1) = f^2(x_1)e_2$ is 1-periodic (in fact $\frac{1}{2}$-periodic) in x_1. ☐

We also have that $\sup\{|u_j^1(x)| : x \in \frac{1}{2}\overline{Q}, j \in \{1, 2\}\} = \frac{1}{2}$. Then

$$\begin{aligned} \nabla u_1^1(x) &= \begin{cases} +\operatorname{sgn}(x_{i(x)})e_{i(x)} & |x_{i(x)}| + |x_{s(x)}| < 1, \\ -\operatorname{sgn}(x_{s(x)})e_{s(x)} & |x_{i(x)}| + |x_{s(x)}| > 1, \end{cases} \\ \nabla u_2^1(x) &= \begin{cases} f^{2\prime}(x_{s(x)})e_{s(x)} & |x_{i(x)}| + |x_{s(x)}| < 1, \\ f^{2\prime}(x_{i(x)})e_{i(x)} & |x_{i(x)}| + |x_{s(x)}| > 1, \end{cases} \end{aligned}$$

since $|f^{2\prime}(t)| = 1$ for $t \notin \frac{1}{4}\mathbb{Z}$, we have $\nabla u^1(x) \in O(2)$ almost everywhere. We now define the function $v : \frac{1}{4}\overline{Q} \to \mathbb{R}$ by

$$\begin{aligned} v_1(x) &= \begin{cases} f^3(|X_s|(x)) & |X_i|(x) + |X_s|(x) \leqslant \frac{1}{2}, \\ f^3(|X_i|(x)) & |X_i|(x) + |X_s|(x) \geqslant \frac{1}{2}, \end{cases} \\ v_2(x) &= \inf\{f^2(|X_i|(x)), f^2(|X_s|(x))\}, \end{aligned}$$

for $x \in Q_s$, and

$$v_1(x) = \begin{cases} f^3(|X_i|(x)) & |X_i|(x) + |X_s|(x) \leqslant \frac{1}{2}, \\ f^3(|X_s|(x)) & |X_i|(x) + |X_s|(x) \geqslant \frac{1}{2}, \end{cases}$$

$$v_2(x) = \sup\{f^2(|X_i|(x)), f^2(|X_s|(x))\},$$

for $x \in Q_i$, where Q_i, Q_s are the sets for which x_2 is the smallest and largest in absolute value in $\{x_1, x_2\}$ respectively. One can verify that v is well-defined and Lipschitz on $\frac{1}{4}Q$.

Proposition 2.2. *The function v has the following properties:*

1. $v(x_1, x_2) = v(|x_1|, |x_2|)$.

2. $v(x_1, x_2) = v(x_2, x_1)$.

Proof. This follows directly from the properties of $|X_i|$ and $|X_s|$. □

We also have that $\sup\{|v_j(x)| : x \in \frac{1}{4}\overline{Q}, j \in \{1, 2\}\} = \frac{1}{2}$. Then

$$\nabla v_1^1(x) = \begin{cases} f^{3\prime}(x_{s(x)})e_{s(x)} & |x_{i(x)}| + |x_{s(x)}| < \frac{1}{2}, \\ f^{3\prime}(x_{i(x)})e_{i(x)} & |x_{i(x)}| + |x_{s(x)}| > \frac{1}{2}, \end{cases}$$

$$\nabla v_2^1(x) = \begin{cases} f^{2\prime}(x_{i(x)})e_{i(x)} & |x_{i(x)}| + |x_{s(x)}| < \frac{1}{2}, \\ f^{2\prime}(x_{s(x)})e_{s(x)} & |x_{i(x)}| + |x_{s(x)}| > \frac{1}{2}, \end{cases}$$

for $x \in Q_s$, and

$$\nabla v_1^1(x) = \begin{cases} f^{3\prime}(x_{i(x)})e_{i(x)} & |x_{i(x)}| + |x_{s(x)}| < \frac{1}{2}, \\ f^{3\prime}(x_{s(x)})e_{s(x)} & |x_{i(x)}| + |x_{s(x)}| > \frac{1}{2}, \end{cases}$$

$$\nabla v_3^1(x) = \begin{cases} f^{2\prime}(x_{s(x)})e_{s(x)} & |x_{i(x)}| + |x_{s(x)}| < \frac{1}{2}, \\ f^{2\prime}(x_{i(x)})e_{i(x)} & |x_{i(x)}| + |x_{s(x)}| > \frac{1}{2}, \end{cases}$$

for $x \in Q_i$, so $\nabla v(x) \in \mathrm{O}(2)$ almost everywhere.

Lemma 2.3. *([8, Claim 2 & 3]) We have the following properties of u^1 and v:*

- $u^1(x_1, 1) = v\left(x_1 \bmod 1 - \frac{1}{2}, 0\right)$.

- *For all $\xi_1 \in \left[-\frac{1}{2}, +\frac{1}{2}\right]$, we have*

$$v_{r-1}(\xi_1, 0) = 2v_r\left(\frac{1}{2}\xi_1 + \frac{1}{4}, \frac{1}{2}\right),$$

where the indices are taken modulo 2.

Proof. For the first part, consider an arbitrary t. We then have

$$v_1(t,0) = \begin{cases} f^3(|t|) & |t| \leqslant \frac{1}{2}, \\ f^3(0) & |t| \geqslant \frac{1}{2}, \end{cases}$$
$$v_2(t,0) = \inf\{f^2(0), f^2(|t|)\},$$

for $(t,0) \in Q_s$, and

$$v_1(t,0) = \begin{cases} f^3(0) & |t| \leqslant \frac{1}{2}, \\ f^3(|t|) & |t| \geqslant \frac{1}{2}, \end{cases}$$
$$v_2(t,0) = \sup\{f^2(0), f^2(|t|)\},$$

for $(t,0) \in Q_i$. We note that

$$f^2(0) = f^3(0) = 0.$$

Taking $t = x_1 \bmod 1 - \frac{1}{2} \in \left[-\frac{1}{2}, +\frac{1}{2}\right]$, then

$$(t,0) \in Q_s^2 \iff t = 0,$$
$$(t,0) \in Q_i^2 \iff t \neq 0,$$

and thus we obtain

$$v_1(t,0) = f^3(t) = 0,$$
$$v_2(t,0) = 0,$$

for $t = 0$, and

$$v_1(t,0) = 0,$$
$$v_2(t,0) = f^2(t),$$

for $t \neq 0$. This can be written more simply as just $v(t,0) = f^2(t)e_2$. The proof then follows from the fact that f^2 is $\frac{1}{2}$-periodic and so

$$f^2(t) = f^2(x_1).$$

Moving on to the second part, we have already determined that $v(\xi_1, 0) = f^2(\xi_1)e_2$, for any $\xi_1 \in \left[-\frac{1}{2}, +\frac{1}{2}\right]$. Also, for any $x_1 \in \left[-\frac{1}{2}, +\frac{1}{2}\right]$, we have

$$v_1\left(x_1, \frac{1}{2}\right) = \begin{cases} f^3\left(\frac{1}{2}\right) & |x_1| = 0, \\ f^3(|x_1|) & |x_1| \geqslant 0, \end{cases}$$
$$v_2\left(x_1, \frac{1}{2}\right) = \inf\left\{f^2(|x_1|), f^2\left(\frac{1}{2}\right)\right\},$$

for $\left(x_1, \frac{1}{2}\right) \in Q_s$, and

$$v_1\left(x_1, \frac{1}{2}\right) = \begin{cases} f^3(|x_1|) & |x_1| = 0, \\ f^3\left(\frac{1}{2}\right) & |x_1| \geqslant 0, \end{cases}$$

$$v_2\left(x_1, \frac{1}{2}\right) = \sup\left\{f^2(|x_1|), f^2\left(\frac{1}{2}\right)\right\},$$

for $\left(x_1, \frac{1}{2}\right) \in Q_i$. We simplify in a similar manner to the previous part and find that $v\left(x_1, \frac{1}{2}\right) = f^3(x_1)e_1$. Now, taking $x_1 = \frac{1}{2}\xi_1 + \frac{1}{4}$, we have

$$v_1\left(\frac{1}{2}\xi_1 + \frac{1}{4}, \frac{1}{2}\right) = f^3\left(\frac{1}{2}\xi_1 + \frac{1}{4}\right) = \frac{1}{2}f^2(\xi_1) = \frac{1}{2}v_2(\xi_1, 0)$$

and $v_2\left(\frac{1}{2}\xi_1 + \frac{1}{4}, \frac{1}{2}\right) = 0 = \frac{1}{2}v_1(\xi_1, 0)$, completing the proof. □

We now define the layering function $\ell^1 : \mathbb{R} \times \left[-\frac{1}{2}, +\frac{1}{2}\right] \to \mathbb{R}^2$, by

$$\ell^1(x_1, x_2) = v\left(x_1 \bmod 1 - \frac{1}{2}, x_2\right).$$

Lemma 2.4. *([8, Claim 4]) We have* $\ell^1(x_1, x_2) = \ell^1(|x_1|, |x_2|)$.

Proof. This follows directly from the properties of v and the fact that

$$\left|x_1 \bmod 1 - \frac{1}{2}\right| = \left||x_1| \bmod 1 - \frac{1}{2}\right|.$$

More specifically, we have

$$\begin{aligned} \ell^1(x_1, x_2) &= v\left(x_1 \bmod 1 - \frac{1}{2}, x_2\right) \\ &= v\left(\left|x_1 \bmod 1 - \frac{1}{2}\right|, |x_2|\right) \\ &= v\left(\left||x_1| \bmod 1 - \frac{1}{2}\right|, |x_2|\right) \\ &= v\left(|x_1| \bmod 1 - \frac{1}{2}, |x_2|\right) \\ &= \ell^1(|x_1|, |x_2|). \end{aligned}$$

□

Furthermore, we define $\ell^n : \mathbb{R} \times \left[-\frac{1}{2^n}, +\frac{1}{2^n}\right] \to \mathbb{R}^2$, by

$$\ell^n(x_1, x_2) = \frac{1}{2^{n-1}}\ell^1\left(2^{n-1}x_1, 2^{n-1}x_2\right)$$

for each $n \in \mathbb{N}$. Then the result of the previous lemma extends to all the ℓ^n.

Lemma 2.5. *([8, Claim 5]) For each $n \in \mathbb{N}$, we have*

$$\ell^{n+1}_{r-1}(x_1, 0) = \ell^n_r\left(x_1, \frac{1}{2^n}\right),$$

where the indices are taken modulo 2.

Proof. We calculate

$$\begin{aligned}\ell^{n+1}(x_1, 0) &= \frac{1}{2^n}\ell^1(2^n x_1, 0)\\ &= \frac{1}{2^n}v\left(2^n x_1 \bmod 1 - \frac{1}{2}, 0\right)\\ &= \frac{1}{2^n}\mathcal{T}\left(2v\left(\frac{1}{2}\left(2^n x_1 \bmod 1 - \frac{1}{2}\right) + \frac{1}{4}, \frac{1}{2}\right)\right),\end{aligned}$$

where $\mathcal{T}$ denotes the linear operator corresponding to the permutation $(1\,2)$. Then

$$\begin{aligned}\ell^{n+1}(x_1, 0) &= \mathcal{T}\left(\frac{1}{2^{n-1}}v\left(\frac{1}{2}\left(2^n x_1 \bmod 1\right), \frac{1}{2}\right)\right)\\ &= \mathcal{T}\left(\frac{1}{2^{n-1}}v\left(2^{n-1}x_1 \bmod 1 - \frac{1}{2}\chi_S(x_1), \frac{1}{2}\right)\right),\end{aligned}$$

where $S = \bigcup_{n\in\mathbb{Z}}[2n-1, 2n)$. Now, since

$$t \mapsto v\left(t, \frac{1}{2}\right) = f^3(t)e_1$$

is $\frac{1}{2}$-periodic, we can conclude that

$$\begin{aligned}\ell^{n+1}(x_1, 0) &= \mathcal{T}\left(\frac{1}{2^{n-1}}v\left(2^{n-1}x_1 \bmod 1 - \frac{1}{2}, \frac{1}{2}\right)\right)\\ &= \mathcal{T}\left(\frac{1}{2^{n-1}}\ell^1\left(2^{n-1}x_1, \frac{1}{2}\right)\right)\\ &= \mathcal{T}\ell^n\left(x_1, \frac{1}{2^n}\right).\end{aligned}$$

□

We now define the layers $\mathcal{L}^n$, for each $n \in \mathbb{N}$, by

$$\begin{aligned}\mathcal{L}^n &= \left\{(x_1, x_2) : \sum_{k=0}^{n-2}\frac{1}{2^k} \leqslant |x_2| \leqslant \sum_{k=0}^{n-1}\frac{1}{2^k} \;\;\&\;\; |x_1| \leqslant |x_2|\right\}\\ &= \left\{(x_1, x_2) : 2 - 2^{2-n} \leqslant |x_2| \leqslant 2 - 2^{1-n} \;\;\&\;\; |x_1| \leqslant |x_2|\right\}.\end{aligned}$$

We have already defined u^1 on $\mathcal{L}^1$. We further define u^n on $\mathcal{L}^n$ by

$$u^n_j(x_1, x_2) = \ell^{n-1}_{j-(n+1)}\left(x_1, |x_2| - \sum_{k=0}^{n-2}\frac{1}{2^k}\right)$$

for each $n \geqslant 2$.

Proposition 2.6. *We have the following property of u^n:*

$$u^n(x_1, x_2) = u^n(|x_1|, |x_2|).$$

Proof. If $(x_1, x_2) \in \mathcal{L}^n$, then $|x_2| - \sum_{k=0}^{n-2} \frac{1}{2^k} \geqslant 0$ so

$$|x_2| - \sum_{k=0}^{n-2} \frac{1}{2^k} = \left| |x_2| - \sum_{k=0}^{n-2} \frac{1}{2^k} \right|.$$

The result then follows from lemma 2.4. □

Lemma 2.7. *([8, Claim 6]) We have continuity from u^{n-1} to u^n, in the sense that*

$$u_j^n\left(x_1, \sum_{k=0}^{n-2} \frac{1}{2^k}\right) = u_j^{n-1}\left(x_1, \sum_{k=0}^{n-2} \frac{1}{2^k}\right) \qquad n \geqslant 2.$$

Proof. We first observe that

$$u_j^n\left(x_1, \sum_{k=0}^{n-2} \frac{1}{2^k}\right) = \ell_{j-(n+1)}^{n-1}\left(x_1, \sum_{k=0}^{n-2} \frac{1}{2^k} - \sum_{k=0}^{n-2} \frac{1}{2^k}\right) = \ell_{j-(n+1)}^{n-1}(x_1, 0)$$

and

$$u_j^{n-1}\left(x_1, \sum_{k=0}^{n-2} \frac{1}{2^k}\right) = \ell_{j-n}^{n-2}\left(x_1, \frac{1}{2^{n-2}}\right).$$

The proof then follows from making the substitution

$$n \mapsto n-2$$
$$r \mapsto j-n$$

in Lemma 2.5. □

We then extend each u^n to the annulus

$$\begin{aligned}\mathcal{A}^n &:= \left\{x : \sum_{k=0}^{n-2} \frac{1}{2^k} \leqslant |x|_\infty \leqslant \sum_{k=0}^{n-1} \frac{1}{2^k}\right\} \\ &= \left\{x : 2 - 2^{2-n} \leqslant |x|_\infty \leqslant 2 - 2^{1-n}\right\}\end{aligned}$$

by setting

$$u^n(x) = u^n(|X_i|(x), |X_s|(x)),$$

which are all continuous. We also have that $\sup\{|u_j^n(x)| : x \in \mathcal{A}^n\} \leqslant \frac{1}{2^{n-1}}$. To compute the gradient ∇u^n at a point $x = \overline{x}$, we recall that there exists integers

$$\bar{\imath} = i(\overline{x}) \quad \overline{s} = s(\overline{x})$$

such that $u^n(x) = u^n(x_{\bar{\imath}}, x_{\bar{s}})$ for all x in a neighbourhood of $\overline{x}$. We then calculate the gradient with respect to $(x_{\bar{\imath}}, x_{\bar{s}})$ to be

$$\begin{aligned}
\nabla_{\bar{\imath},\bar{s}} u^n_j(x_{\bar{\imath}}, x_{\bar{s}}) &= \nabla_{\bar{\imath},\bar{s}} \ell^{n-1}_{j-(n+1)} \left(x_{\bar{\imath}}, |x_{\bar{s}}| - \sum_{k=0}^{n-2} \frac{1}{2^k} \right) \\
&= \nabla_{\bar{\imath},\bar{s}} \frac{1}{2^{n-2}} \ell^1_{j-(n+1)} \left(2^{n-2} x_{\bar{\imath}}, 2^{n-2} \left(|x_{\bar{s}}| - \sum_{k=0}^{n-2} \frac{1}{2^k} \right) \right) \\
&= \nabla_{\bar{\imath},\bar{s}} \frac{1}{2^{n-2}} v_{j-(n+1)} \left(2^{n-2} x_{\bar{\imath}} \bmod 1, 2^{n-2} \left(|x_{\bar{s}}| - \sum_{k=0}^{n-2} \frac{1}{2^k} \right) \right) \\
&= \nabla_{\bar{\imath},\bar{s}} v_{j-(n+1)}(\xi_{\bar{\imath}}, \xi_{\bar{s}}) \cdot \Sigma
\end{aligned}$$

almost everywhere by chain rule. Here

$$(\xi_{\bar{\imath}}, \xi_{\bar{s}}) = \left(2^{n-2} x_{\bar{\imath}} \bmod 1, 2^{n-2} \left(|x_{\bar{s}}| - \sum_{k=0}^{n-2} \frac{1}{2^k} \right) \right), \quad \Sigma = \begin{pmatrix} 1 & 0 \\ 0 & \operatorname{sgn}(x_{\bar{s}}) \end{pmatrix}.$$

Then there is a permutation matrix P (depending only on n) such that

$$\nabla_{\bar{\imath},\bar{s}} u^n(x_{\bar{\imath}}, x_{\bar{s}}) = \nabla_{\bar{\imath},\bar{s}} v(\xi_{\bar{\imath}}, \xi_{\bar{s}}) \cdot P\Sigma \in \mathrm{O}(2).$$

We define $u^* : Q \to \mathbb{R}^2$ by

$$u^*(x) = u^n(x) \quad x \in \mathcal{A}^n \quad n = 1, 2, \ldots$$

Then u^* is 1-Lipschitz and solves 1.1.

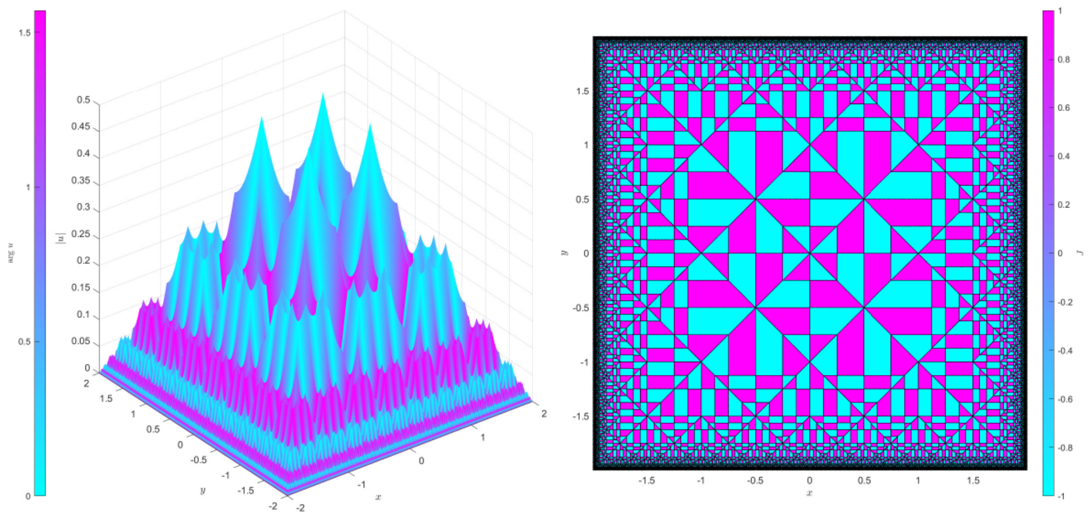


Figure 2.1: Plot of particular solution u^* and its Jacobian $\det \nabla u^*$.

2.2. Simplifications of Constructed Solution

Since we are working in two dimensions and the previously constructed solution has been adapted from the three dimensional case, there are a number of natural simplifications that occur. Firstly, an equivalent form for $u^1 : \frac{1}{2}\overline{Q} \to \mathbb{R}^2$ is

$$u_1^1(x) = \inf\{f^1(|x_1|), f^1(|x_2|)\},$$
$$u_2^1(x) = \begin{cases} f^2(|X_s|(x)) & |x|_1 \leqslant 1, \\ f^2(|X_i|(x)) & |x|_1 \geqslant 1. \end{cases}$$

For $v : \frac{1}{4}\overline{Q} \to \mathbb{R}$, we have

$$v_1(x) = \begin{cases} f^3(|x_2|) & ||x||_1 \leqslant \frac{1}{2}, \\ f^3(|x_1|) & ||x||_1 \geqslant \frac{1}{2}, \end{cases}$$
$$v_2(x) = \begin{cases} \inf\{f^2(|x_1|), f^2(|x_2|)\} & |x_1| \leqslant |x_2|, \\ \sup\{f^2(|x_1|), f^2(|x_2|)\} & |x_1| \geqslant |x_2|. \end{cases}$$

We can expand to get

$$u_1^1(x) = \begin{cases} f^1(|x_1|) & f^1(|x_1|) \leqslant f^1(|x_2|), \\ f^1(|x_2|) & f^1(|x_1|) \geqslant f^1(|x_2|), \end{cases}$$
$$u_2^1(x) = \begin{cases} f^2(|x_1|) & |x|_1 \leqslant 1 \;\;\&\;\; |x_1| \geqslant |x_2| \parallel |x|_1 \geqslant 1 \;\;\&\;\; |x_1| \leqslant |x_2|, \\ f^2(|x_2|) & |x|_1 \leqslant 1 \;\;\&\;\; |x_1| \leqslant |x_2| \parallel |x|_1 \geqslant 1 \;\;\&\;\; |x_1| \geqslant |x_2|, \end{cases}$$
$$v_1(x) = \begin{cases} f^3(|x_1|) & |x|_1 \geqslant \frac{1}{2}, \\ f^3(|x_2|) & |x|_1 \leqslant \frac{1}{2}, \end{cases}$$
$$v_2(x) = \begin{cases} f^2(|x_1|) & f^2(|x_1|) \leqslant f^2(|x_2|) \;\;\&\;\; |x_1| \leqslant |x_2| \parallel f^2(|x_1|) \geqslant f^2(|x_1|) \;\;\&\;\; |x_1| \geqslant |x_2|, \\ f^2(|x_2|) & f^2(|x_1|) \leqslant f^2(|x_2|) \;\;\&\;\; |x_1| \geqslant |x_2| \parallel f^2(|x_1|) \geqslant f^2(|x_1|) \;\;\&\;\; |x_1| \leqslant |x_2|. \end{cases}$$

Labelling the conditions from 1 to 8, one can verify that

$$1 \iff 4,$$
$$2 \iff 3,$$
$$5 \iff 8,$$
$$6 \iff 7.$$

Since the f^k are all even functions, we can drop most of the absolute values and simplify as follows:

$$u^1(x) = \begin{cases} \left(f^1(x_1), f^2(x_2)\right)^\mathsf{T} & |x|_1 \leqslant 1 \ \& \ |x_1| \leqslant |x_2| \parallel |x|_1 \geqslant 1 \ \& \ |x_1| \geqslant |x_2|, \\ \left(f^1(x_2), f^2(x_1)\right)^\mathsf{T} & |x|_1 \leqslant 1 \ \& \ |x_1| \geqslant |x_2| \parallel |x|_1 \geqslant 1 \ \& \ |x_1| \leqslant |x_2|, \end{cases}$$

$$v(x) = \begin{cases} \left(f^3(x_1), f^2(x_2)\right)^\mathsf{T} & |x|_1 \geqslant \frac{1}{2}, \\ \left(f^3(x_2), f^2(x_1)\right)^\mathsf{T} & |x|_1 \leqslant \frac{1}{2}. \end{cases}$$

If we now assume $x \in \mathcal{L} := \bigcup_{n=1}^{\infty} \mathcal{L}^n$, we can simply write

$$u^1(x) = \begin{cases} \left(f^1(x_1), f^2(x_2)\right)^\mathsf{T} & |x|_1 \leqslant 1, \\ \left(f^1(x_2), f^2(x_1)\right)^\mathsf{T} & |x_1| \geqslant 1, \end{cases}$$

$$v(x) = \begin{cases} \left(f^3(x_1), f^2(x_2)\right)^\mathsf{T} & |x|_1 \geqslant \frac{1}{2}, \\ \left(f^3(x_2), f^2(x_1)\right)^\mathsf{T} & |x|_1 \leqslant \frac{1}{2}. \end{cases}$$

We then calculate the gradients to be

$$\nabla u^1(x) = \begin{cases} \begin{pmatrix} f^{1\prime}(x_1) & 0 \\ 0 & f^{2\prime}(x_2) \end{pmatrix} & |x|_1 \leqslant 1, \\ \begin{pmatrix} 0 & f^{1\prime}(x_2) \\ f^{2\prime}(x_1) & 0 \end{pmatrix} & |x_1| \geqslant 1, \end{cases}$$

$$\nabla v(x) = \begin{cases} \begin{pmatrix} f^{3\prime}(x_1) & 0 \\ 0 & f^{2\prime}(x_2) \end{pmatrix} & |x|_1 \geqslant \frac{1}{2}, \\ \begin{pmatrix} 0 & f^{3\prime}(x_2) \\ f^{2\prime}(x_1) & 0 \end{pmatrix} & |x|_1 \leqslant \frac{1}{2}. \end{cases}$$

Since $|f^{k\prime}| = 1$ almost everywhere, the gradients are both in $\mathrm{O}(2)$ almost everywhere.
Then we define $\ell^n : \mathbb{R} \times \left[-\frac{1}{2^n}, +\frac{1}{2^n}\right] \to \mathbb{R}^2$ by

$$\ell^1(x_1, x_2) = v\left(x_1 \bmod 1 - \frac{1}{2}, x_2\right)$$
$$\ell^n(x_1, x_2) = \frac{1}{2^{n-1}} \ell^1(2^{n-1}x_1, 2^{n-1}x_2)$$

and $u^n : \mathcal{L}^n \to \mathbb{R}^2$ (for $n > 1$) by

$$u^n(x_1, x_2) = \mathcal{T}^{n-1}\ell^{n-1}\left(x_1, |x_2| - 2 + 2^{2-n}\right) \qquad \mathcal{T} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Then the gradients are

$$\nabla \ell^n(x_1, x_2) = \nabla \ell^1(2^{n-1}x_1, 2^{n-1}x_2) = \nabla v\left(2^{n-1}x_1 \bmod 1 - \frac{1}{2}, 2^{n-1}x_2\right)$$

and

$$\begin{aligned}\nabla u^n(x_1, x_2) &= \mathcal{T}^{n-1}\nabla \ell^{n-1}\left(x_1, |x_2| - 2 + 2^{2-n}\right)\begin{pmatrix}1 & 0\\ 0 & \operatorname{sgn}(x_2)\end{pmatrix}\\ &= \mathcal{T}^{n-1}\nabla v\left(2^{n-2}x_1 \bmod 1 - \frac{1}{2}, 2^{n-2}|x_2| - 2^{n-1} + 1\right)\begin{pmatrix}1 & 0\\ 0 & \operatorname{sgn}(x_2)\end{pmatrix}.\end{aligned}$$

Hence

$$\begin{aligned}\det \nabla u^1(x_1, x_2) &= \begin{cases} +f^{1\prime}(x_1)f^{2\prime}(x_2) & |x|_1 \leqslant 1,\\ -f^{2\prime}(x_1)f^{1\prime}(x_2) & |x|_1 \geqslant 1,\end{cases}\\ \det \nabla u^n(x_1, x_2) &= (-1)^{n-1}\operatorname{sgn}(x_2)\det \nabla v\left(2^{n-2}x_1 \bmod 1 - \frac{1}{2}, 2^{n-2}|x_2| - 2^{n-1} + 1\right)\\ &= \begin{cases} +(-1)^{n-1}\operatorname{sgn}(x_2)f^{3\prime}(y_1)f^{2\prime}(y_2) & |y|_1 \geqslant \frac{1}{2},\\ -(-1)^{n-1}\operatorname{sgn}(x_2)f^{2\prime}(y_1)f^{3\prime}(y_2) & |y|_1 \leqslant \frac{1}{2},\end{cases}\end{aligned}$$

where

$$y(x) = \left(2^{n-2}x_1 \bmod 1 - \frac{1}{2}, 2^{n-2}|x_2| - 2^{n-1} + 1\right)^{\mathsf{T}}.$$

This allows us to calculate the Jacobian on $\mathcal{L}$. To get the Jacobian on the rest of Q, we use

$$\nabla(u^*(|X_i|(x), |X_s|(x))) = \begin{cases} \nabla u^*(|X_i|(x), |X_s|(x)) \cdot \begin{pmatrix}\operatorname{sgn}(x_1) & 0\\ 0 & \operatorname{sgn}(x_2)\end{pmatrix} & |x_1| \leqslant |x_2|,\\ \nabla u^*(|X_i|(x), |X_s|(x)) \cdot \begin{pmatrix}0 & \operatorname{sgn}(x_2)\\ \operatorname{sgn}(x_1) & 0\end{pmatrix} & |x_1| \geqslant |x_2|,\end{cases}$$

so

$$\det \nabla u^*(x) = \begin{cases} +\det \nabla u^n(|x_1|, |x_2|) \cdot \operatorname{sgn}(x_1)\operatorname{sgn}(x_2) & x \in \mathcal{L}^n,\\ -\det \nabla u^n(|x_2|, |x_1|) \cdot \operatorname{sgn}(x_1)\operatorname{sgn}(x_2) & x \in \mathcal{A}^n - \mathcal{L}^n.\end{cases}$$

2.3. Algorithmic Approach to Constructing General Solutions

We will now seek to generalise this construction to find more solutions and/or solutions meeting a different boundary condition. A natural approach is to find a way of splitting up the domain into discrete regions (triangles, for example) and then to design a procedure of going through these regions and assigning a value of the Jacobian $\det \nabla u$ to each one. This should be done in such a way that the following two properties are achieved.

- The solution u is continuous.
- The boundary condition is met.

2.3.1. Properties of Rotation Matrices

We shall first state some properties of rotation matrices, that will be useful for some of the calculations performed in this section.

Proposition 2.8. *For any $\theta \in \mathbb{R}$ and $\kappa \in \mathbb{Z}$, we have $R(\theta)^\kappa = R(\kappa\theta)$.*

Proposition 2.9. *For any matrix $R_0 \in \mathrm{SO}(2)$, we have:*

1. $R_0 I_\alpha = I_\alpha R_0^\alpha$.
2. $R(R_0 I_\alpha a) = \alpha R_0 R(a)^\alpha$ *for every* $a \in \mathbb{R}^2$.
3. $\alpha R(\alpha a) = R(a)$ *for every* $a \in \mathbb{R}^2$.

for each $\alpha \in \{-1, +1\}$.

We then get the following result:

Lemma 2.10. *Given two vectors $a, b \in \mathbb{R}^2$, with $|a|_2 = |b|_2$, there are exactly two matrices $R \in \mathrm{O}(2)$ satisfying $Ra = b$. They are given by*

$$R_\alpha(a, b) = \alpha R(\theta(b) - \alpha\theta(a)) I_\alpha = \alpha R(b) I_\alpha R(a)^\mathsf{T} \in \mathrm{SO}(2) I_\alpha \qquad \alpha \in \{-1, +1\}.$$

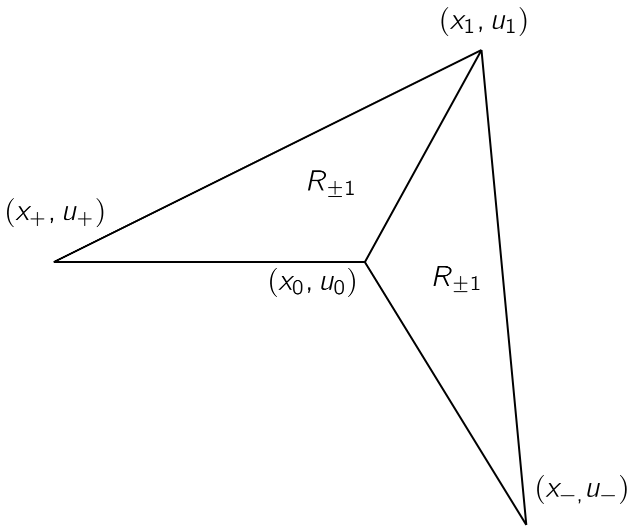
The proofs for the two propositions are purely algebraic manipulations. The lemma relies on the assertion that there is just one reflection and one rotation of a vector onto another in two dimensions. This is quite an intuitive geometric result.

2.3.2. Implications of Continuity

We start by considering four vertices x_0, x_1, x_+, x_- forming two triangles in the plane and prescribe $u(x_0) = u_0$, $u(x_1) = u_1$ with $|u_1 - u_0| = |x_1 - x_0|$. Using lemma 2.10, we have two choices of gradient for each triangle

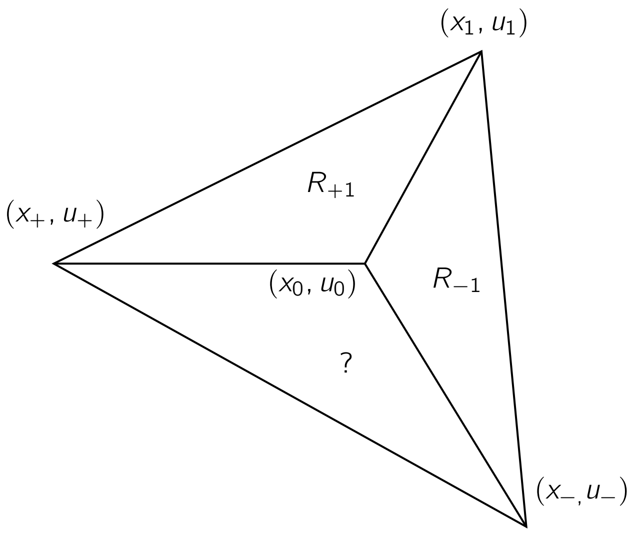
$$R_\alpha := R_\alpha(x_1 - x_0, u_1 - u_0) \qquad \alpha \in \{-1, +1\}.$$

These choices will then determine the $u_\pm = u(x_\pm)$.



Now consider four vertices x_0, x_1, x_+, x_- forming three triangles in the plane and prescribe $u(x_0) = u_0$, $u(x_1) = u_1$ with $|u_1 - u_0| = |x_1 - x_0|$. Using lemma 2.10, we have two choices of gradient for the triangles $\Delta x_0 x_1 x_\pm$

$$R_\alpha := R_\alpha(x_1 - x_0, u_1 - u_0) \qquad \alpha \in \{-1, +1\}.$$



If we choose the same gradient for each of the triangles, it then follows that the final triangle must also have this gradient. Hence we shall pick one of each gradient and, for simplicity, we shall take R_{+1} on $\triangle x_0 x_1 x_+$ and R_{-1} on $\triangle x_0 x_1 x_-$. This determines the $u_\pm = u(x_\pm)$. Now we consider the triangle $\triangle x_0 x_+ x_-$.
By considering the edge from (x_0, u_0) to (x_+, u_+) we see that, if we want a gradient in $\mathrm{SO}(2)I_{+1}$ on the final triangle, it must be R_{+1}.
By considering the edge from (x_0, u_0) to (x_-, u_-) we see that, if we want a gradient in $\mathrm{SO}(2)I_{-1}$ on the final triangle, it must be R_{-1}.
However, in general,

$$u_+ - u_- \neq R_\alpha(x_+ - x_-) \qquad \alpha \in \{-1, +1\}.$$

Thus, neither of these choices are guaranteed to give a continuous solution u.

2.3.3. Deriving a Continuity Equation

We will now consider a sequence of edges $\vec{x}_i \in \mathbb{R}^2$ in the independent variable and a sequence of choices for the sign of the gradient $\alpha_i \in \{-1, +1\}$. Given a $\vec{u}_0 \in \mathbb{R}^2$ such that $|\vec{x}_0|_2 = |\vec{u}_0|_2$, we generate a sequence of edges $\vec{u}_i$ in the dependent variable via

$$\vec{u}_i = R_{\alpha_i}(\vec{x}_{i-1}, \vec{u}_{i-1})\vec{x}_i \qquad i = 1, 2, \ldots$$

Proposition 2.11. *For any $n \in \mathbb{N}$, we have*

$$\vec{u}_n = \alpha_n R(\vec{u}_0) R(\alpha_1 \theta(\vec{x}_0))^\mathsf{T} \prod_{i=1}^{n-1} R((\alpha_i - \alpha_{i+1})\theta(\vec{x}_i)) I_{\alpha_n} \vec{x}_n.$$

Proof. We shall prove the statement by induction. When $n = 1$, we obtain

$$\begin{aligned} \vec{u}_1 &= R_{\alpha_1}(\vec{x}_0, \vec{u}_0)\vec{x}_1 \\ &= \alpha_1 R(\vec{u}_0) I_{\alpha_1} R(\vec{x}_0)^{-1} \vec{x}_1 \\ &= \alpha_1 R(\vec{u}_0) R(\vec{x}_0)^{-\alpha_1} I_{\alpha_1} \vec{x}_1 \\ &= \alpha_1 R(\vec{u}_0) R(\alpha_1 \vec{x}_0)^\mathsf{T} I_{\alpha_1} \vec{x}_1, \end{aligned}$$

proving the $n = 1$ case. Now suppose the statement holds for $n = k \in \mathbb{N}$. Then setting $n = k + 1$, we have

$$\begin{aligned}
\vec{u}_{k+1} &= R_{\alpha_{k+1}}(\vec{x}_k, \vec{u}_k)\vec{x}_{k+1} \\
&= \alpha_{k+1} R(\vec{u}_k) I_{\alpha_{k+1}} R(\vec{x}_k)^{\mathsf{T}} \vec{x}_{k+1} \\
&= \alpha_{k+1} R\left(\alpha_k R(\vec{u}_0) R(\alpha_1\theta(\vec{x}_0))^{\mathsf{T}} \prod_{i=1}^{k-1} R((\alpha_i - \alpha_{i+1})\theta(\vec{x}_i)) I_{\alpha_k}\vec{x}_k\right) I_{\alpha_{k+1}} R(\vec{x}_k)^{\mathsf{T}} \vec{x}_{k+1} \\
&= \alpha_{k+1} R\left(R(\vec{u}_0) R(\alpha_1\theta(\vec{x}_0))^{\mathsf{T}} \prod_{i=1}^{k-1} R((\alpha_i - \alpha_{i+1})\theta(\vec{x}_i)) I_{\alpha_k}\alpha_k\vec{x}_k\right) R(\vec{x}_k)^{-\alpha_{k+1}} I_{\alpha_{k+1}} \vec{x}_{k+1} \\
&= \alpha_{k+1}\alpha_k R(\vec{u}_0) R(\alpha_1\theta(\vec{x}_0))^{\mathsf{T}} \prod_{i=1}^{k-1} R((\alpha_i - \alpha_{i+1})\theta(\vec{x}_i)) R(\alpha_k\vec{x}_k)^{\alpha_k} R(\vec{x}_k)^{-\alpha_{k+1}} I_{\alpha_{k+1}} \vec{x}_{k+1} \\
&= \alpha_{k+1} R(\vec{u}_0) R(\alpha_1\theta(\vec{x}_0))^{\mathsf{T}} \prod_{i=1}^{k-1} R((\alpha_i - \alpha_{i+1})\theta(\vec{x}_i)) \left(\alpha_k R(\alpha_k\vec{x}_k)\right)^{\alpha_k} R(\vec{x}_k)^{-\alpha_{k+1}} I_{\alpha_{k+1}} \vec{x}_{k+1} \\
&= \alpha_{k+1} R(\vec{u}_0) R(\alpha_1\theta(\vec{x}_0))^{\mathsf{T}} \prod_{i=1}^{k-1} R((\alpha_i - \alpha_{i+1})\theta(\vec{x}_i)) R(\vec{x}_k)^{\alpha_k} R(\vec{x}_k)^{-\alpha_{k+1}} I_{\alpha_{k+1}} \vec{x}_{k+1} \\
&= \alpha_{k+1} R(\vec{u}_0) R(\alpha_1\theta(\vec{x}_0))^{\mathsf{T}} \prod_{i=1}^{k-1} R((\alpha_i - \alpha_{i+1})\theta(\vec{x}_i)) R((\alpha_k - \alpha_{k+1})\theta(\vec{x}_k)) I_{\alpha_{k+1}} \vec{x}_{k+1} \\
&= \alpha_{k+1} R(\vec{u}_0) R(\alpha_1\theta(\vec{x}_0))^{\mathsf{T}} \prod_{i=1}^{k} R((\alpha_i - \alpha_{i+1})\theta(\vec{x}_i)) I_{\alpha_{k+1}} \vec{x}_{k+1},
\end{aligned}$$

so the statement holds for $n = k + 1$. □

Now we suppose there is $n \in \mathbb{N}$ such that $\vec{x}_n = \vec{x}_0$. For the constructed map to be continuous, we must then have $\vec{u}_n = \vec{u}_0$. Using the previous proposition, we then require

$$\alpha_n R(\vec{u}_0) R(\alpha_1\theta(\vec{x}_0))^{\mathsf{T}} \prod_{i=1}^{n-1} R((\alpha_i - \alpha_{i+1})\theta(\vec{x}_i)) I_{\alpha_n}\vec{x}_n = \vec{u}_0.$$

We can then write $\vec{x}_n = R(\vec{x}_0)e_1$ and $\vec{u}_0 = R(\vec{u}_0)e_1$ to obtain

$$\alpha_n R(\vec{u}_0) R(\alpha_1\theta(\vec{x}_0))^{\mathsf{T}} \prod_{i=1}^{n-1} R((\alpha_i - \alpha_{i+1})\theta(\vec{x}_i)) I_{\alpha_n} R(\vec{x}_0) e_1 = R(\vec{u}_0) e_1.$$

By making some slight simplifications, we find that

$$\alpha_n R(\vec{x}_0)^{\alpha_n} R(\alpha_1\theta(\vec{x}_0))^{\mathsf{T}} \prod_{i=1}^{n-1} R((\alpha_i - \alpha_{i+1})\theta(\vec{x}_i)) I_{\alpha_n} e_1 = e_1$$

and hence

$$\alpha_0 \prod_{i=0}^{n-1} R((\alpha_i - \alpha_{i+1})\theta(\vec{x}_i)) I_{\alpha_0} e_1 = \alpha_0 R\left(\sum_{i=0}^{n-1}(\alpha_i - \alpha_{i+1})\theta(\vec{x}_i)\right) I_{\alpha_0} e_1 = e_1$$

with $\alpha_0 := \alpha_n$. Using lemma 2.10, we must have that

$$\alpha_0 R\left(\sum_{i=0}^{n-1}(\alpha_i - \alpha_{i+1})\theta(\vec{x}_i)\right) I_{\alpha_0} = R_{\alpha_0}(e_1, e_1) = \alpha_0 I_{\alpha_0}.$$

Thus

$$R\left(\sum_{i=0}^{n-1}(\alpha_i - \alpha_{i+1})\theta(\vec{x}_i)\right) = I \Rightarrow \sum_{i=0}^{n-1}(\alpha_i - \alpha_{i+1})\theta(\vec{x}_i) \equiv_{2\pi} 0.$$

Writing $\Delta\alpha_i = \alpha_{i+1} - \alpha_i$, we have found that

$$\sum_{i=0}^{n-1} \Delta\alpha_i \theta(\vec{x}_i) \equiv_{2\pi} 0. \tag{2.1}$$

If we shift indices and rearrange, this can be written equivalently as

$$\sum_{i=1}^{n} \alpha_i \Delta\theta(\vec{x}_{i-1}) \equiv_{2\pi} 0, \tag{2.2}$$

where $\Delta\theta(\vec{x}_i) := \theta(\vec{x}_{i+1}) - \theta(\vec{x}_i)$.

The problem has now become finding a partitioning of the domain Ω (which can be thought of as a set of edges $\vec{X}$) such that, given any cyclic sequence $(\vec{x}_i)_{i=0}^{n} \subset \vec{X}$, we can pick a tuple $\alpha \in \{-1, +1\}^n$ satisfying 2.2. This is clearly a very difficult problem and it is unclear on what the best strategy to solve it is. With this in mind, we shall now seek a different approach that ensures continuity throughout.

2.4. Analytic Approach to Constructing General Solutions

In this section, we shall assume Ω is convex. We shall first derive a method to construct solutions to the general PDI

$$\nabla u(x) \in \mathrm{O}(d) \qquad \text{a.e. } x \in \Omega. \tag{2.3}$$

We can write this PDI as a nonlinear system of PDEs

$$\nabla u_i \cdot \nabla u_j = \delta_{ij} \qquad \forall i, j.$$

In two dimensions, we can isolate the gradients as

$$\begin{aligned}\nabla u_1(x) &= \sigma_1(x) \cdot (+\cos\theta(x), -\sin\theta(x)),\\ \nabla u_2(x) &= \sigma_2(x) \cdot (+\sin\theta(x), +\cos\theta(x)),\end{aligned}$$

where we assume $\sigma_i : \Omega \to \{-1, +1\}$ and $\theta : \Omega \to S^1$ to be piecewise C^1. By taking the curl, we derive the necessary condition

$$\begin{aligned}0 &= \sigma_1(x) \cdot (\cos\theta(x)\theta_x - \sin\theta(x)\theta_y)\\ 0 &= \sigma_2(x) \cdot (\sin\theta(x)\theta_x + \cos\theta(x)\theta_y)\end{aligned}$$

or equivalently

$$R(\theta)\nabla^{\mathsf{T}}\theta = 0 \qquad \text{a.e. in } \Omega.$$

Since $\ker R(\theta) = \{0\}$, we have in fact that θ is piecewise constant. To calculate u we take a line integral from a fixed point $\overline{x} \in \Omega$ to the variable point x along a path that only varies in one coordinate at a time (this will allow us to make use of the ACL property of Sobolev spaces). For example:

$$\gamma(t) = \begin{cases} t(x_1, \overline{x}_2)^{\mathsf{T}} + (1-t)\overline{x} & t \in [0, 1],\\ (t-1)(x_1, x_2)^{\mathsf{T}} + (2-t)(x_1, \overline{x}_2)^{\mathsf{T}} & t \in [1, 2].\end{cases}$$

Then we have that

$$\begin{aligned}u_1(x) &= \overline{u}_1 + \int_{\overline{x}}^{x} \sigma_1(x') \left(+\cos\theta(x'), -\sin\theta(x')\right) \cdot \mathrm{d}x'\\ u_2(x) &= \overline{u}_2 + \int_{\overline{x}}^{x} \sigma_2(x') \left(+\sin\theta(x'), +\cos\theta(x')\right) \cdot \mathrm{d}x'\end{aligned}$$

or equivalently

$$u(x) = \overline{u} + \int_{\overline{x}}^{x} \Sigma(x')R(\theta(x'))\,\mathrm{d}x' \qquad \Sigma(x) := \mathrm{diag}(\sigma_1(x), \sigma_2(x)).$$

This gives us a way of generating a broad class of solutions to 2.3 with $u(\overline{x}) = \overline{u}$.

Now consider a constructed solution $u_0 : \Omega \to \Omega$ to 1.1 (e.g: u^* constructed in section 2.1.) and suppose we wish to find transformations that can create additional solutions u to 1.1.

Definition 2.12. *A function $w \in W^{1,\infty}(\Omega; \Omega)$ is a* **post-transform** *if it satisfies*

$$\nabla w(x) \in \mathrm{O}(d) \qquad a.e.\ x \in \Omega,$$
$$w(0) = 0.$$

We can use the previously derived techniques to find a broad class of post-transforms. Taking $\overline{x} = 0$ and $\overline{u} = 0$ we have that

$$w(x) = \int_0^x \Sigma(x')R(\theta(x'))\,\mathrm{d}\gamma \qquad \Sigma(x) = \mathrm{diag}(\sigma_1(x), \sigma_2(x))$$

will be a post-transforms for all appropriate choices of σ_i and θ.

Definition 2.13. *A function $w \in W^{1,\infty}(\Omega; \Omega)$ is a* **pre-transform** *if it satisfies*

$$\nabla w(x) \in \mathrm{O}(d) \qquad a.e.\ x \in \Omega,$$
$$w(\partial\Omega) \subset \partial\Omega.$$

We can construct piecewise affine pre-transforms in two dimensions intuitively as sequences of rotations, reflections and folds of a piece of paper that represents the domain Ω.

Example 2.14. *The identity transform, given by $w(x) = x$, is both a pre-transform and a post-transform.*

We can now use these transforms to generate more solutions to 1.1

Theorem 2.15. *Let $u_0 \in W^{1,2}(\Omega; \mathbb{R}^2)$ solve 1.1. For any pre-transform w_{pre} and post-transform w_{post}, the composition $u = w_{post} \circ u_0 \circ w_{pre}$ solves 1.1.*

Note that we may have $u \notin W^{1,2}(\Omega; \mathbb{R}^2)$. However, we do know that $w_{\mathrm{post}} \circ u_0 \in W^{1,2}(\Omega; \mathbb{R}^2)$ if $w_{\mathrm{post}} \circ u_0 \in L^2(\Omega; \mathbb{R}^2)$ [20].

Example 2.16. *We shall construct a pre-transform by first folding the domain down the line $x_1 = 0$ and then folding along the line $x_1 = \nu$ for some arbitrary $\nu \in [0, 1]$. We then construct a post-transform using the previously mentioned integral formula with $\theta \equiv 0$, $\sigma_2 \equiv +1$ and*

$$\sigma_1(x) = \begin{cases} +1 & |x|_1 \leqslant 0.1 \pmod{0.2}, \\ -1 & |x|_1 > 0.1 \pmod{0.2}. \end{cases}$$

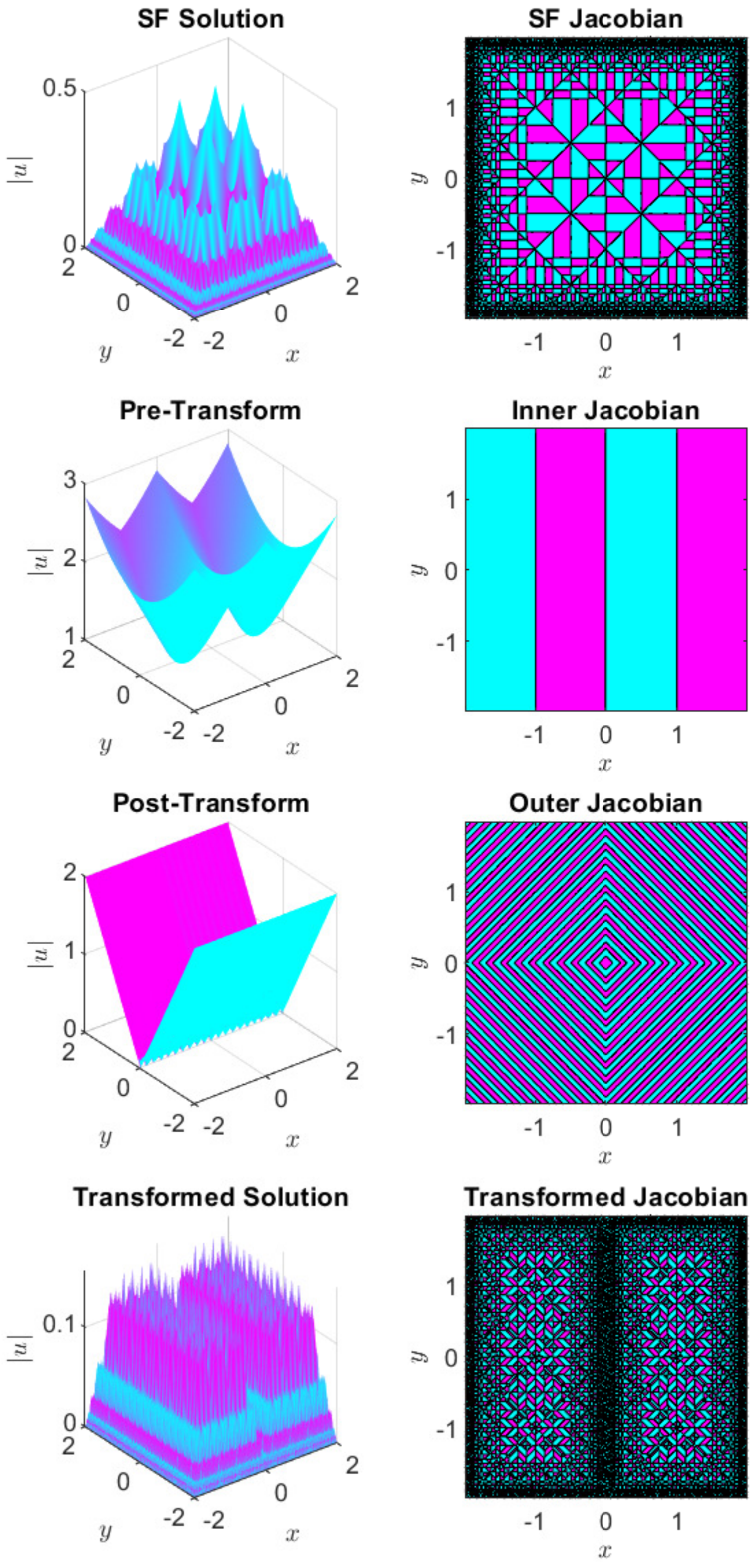


Figure 2.2: Effects of the transformation described in 2.16 with $\nu = 1$.

2.5. Numerical Results

Now that we have a solution u^* to 1.1, we can return to our calculation of I_0 from 1.3.. Furthermore, we can approximate I_0 by using a symbolic computation engine (such as Mathematica) to evaluate the integral on the annuli $\mathcal{A}^n$ for a finite number of n and then bounding the integral on the remainder of the domain. If we take the first n annuli, we get a bound on the error given by

$$\epsilon \leqslant \left| \Omega - \bigcup_{k=1}^{n} \mathcal{A}^k \right| = 4^2 - (4 - 2^{2-n})^2$$

per square. We will use 22 annuli and thus have an error bound of $\epsilon \leqslant 7.62939 \times 10^{-6}$ per square.

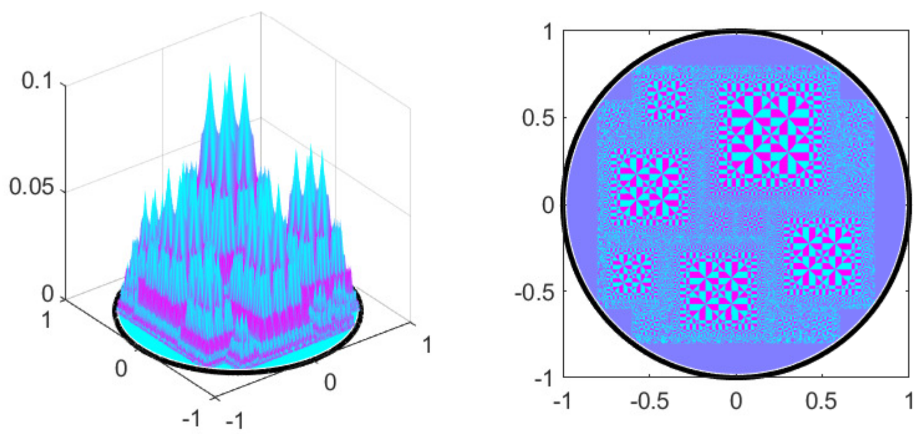


Figure 2.3: Example of a partial tiling of the ball with 15 squares.

Running this calculation with an arrangement of 15 squares, we obtain $I_0 \approx -5.8995 \times 10^{-4}$. For approximating the bound for λ, we take the worst-case scenario

$$|I_0| \geqslant 5.8995 \times 10^{-4} - 15 \times 7.62939 \times 10^{-6} = 4.75513 \times 10^{-4}.$$

This then gives us that $\lambda \leqslant 10094.4$. It is important that we do not use a square tiling with symmetries matching those of u^* as this will immediately give $I_0 = 0$. If instead we calculate the integral with a simpler arrangement of just 2 squares (with centres $(+0.1, +0.3)$, $(-0.3, -0.5)$ and widths 1.0, 0.6, respectively), we obtain $I_0 \approx 1.83384 \times 10^{-2}$. We again take the worst-case scenario

$$|I_0| \geqslant 1.83384 \times 10^{-2} - 2 \times 7.62939 \times 10^{-6} = 1.83231 \times 10^{-2}.$$

This now gives us that $\lambda \leqslant 148.446$.

We could seek to make $|I_0|$ larger by switching out u^* for a different solution to 1.1. This could be done, for example, by applying the transforms mentioned in 2.4.. However these will often lead to a more homogenised partition $Q^\pm$ of Q, which will cause $|I_0| \to 0$. To increase $|I_0|$, it then seems necessary for the partition $Q^\pm$ to be as unhomogenised as possible. In other words, we want to maximise the area of the regions on which u^* is affine. We currently have no techniques to do this reliably.

3 Conclusions and Outlook

To summarise we have found a sufficient condition of the form $\lambda \leqslant \lambda_{\text{suf}}$ and a necessary condition of the form $\lambda \leqslant \lambda_{\text{nec}}$ for the functional

$$E_f(u) = \int_B |\nabla u|_F^2 + f(x) \det \nabla u \, dx \qquad f(x) = \lambda \, |x|_2$$

to have 0 as its global minimum on $W_0^{1,2}(B; \mathbb{R}^2)$. In particular, we have

$$\lambda_{\text{suf}} \approx 9.619, \qquad \lambda_{\text{nec}} \approx 148.446.$$

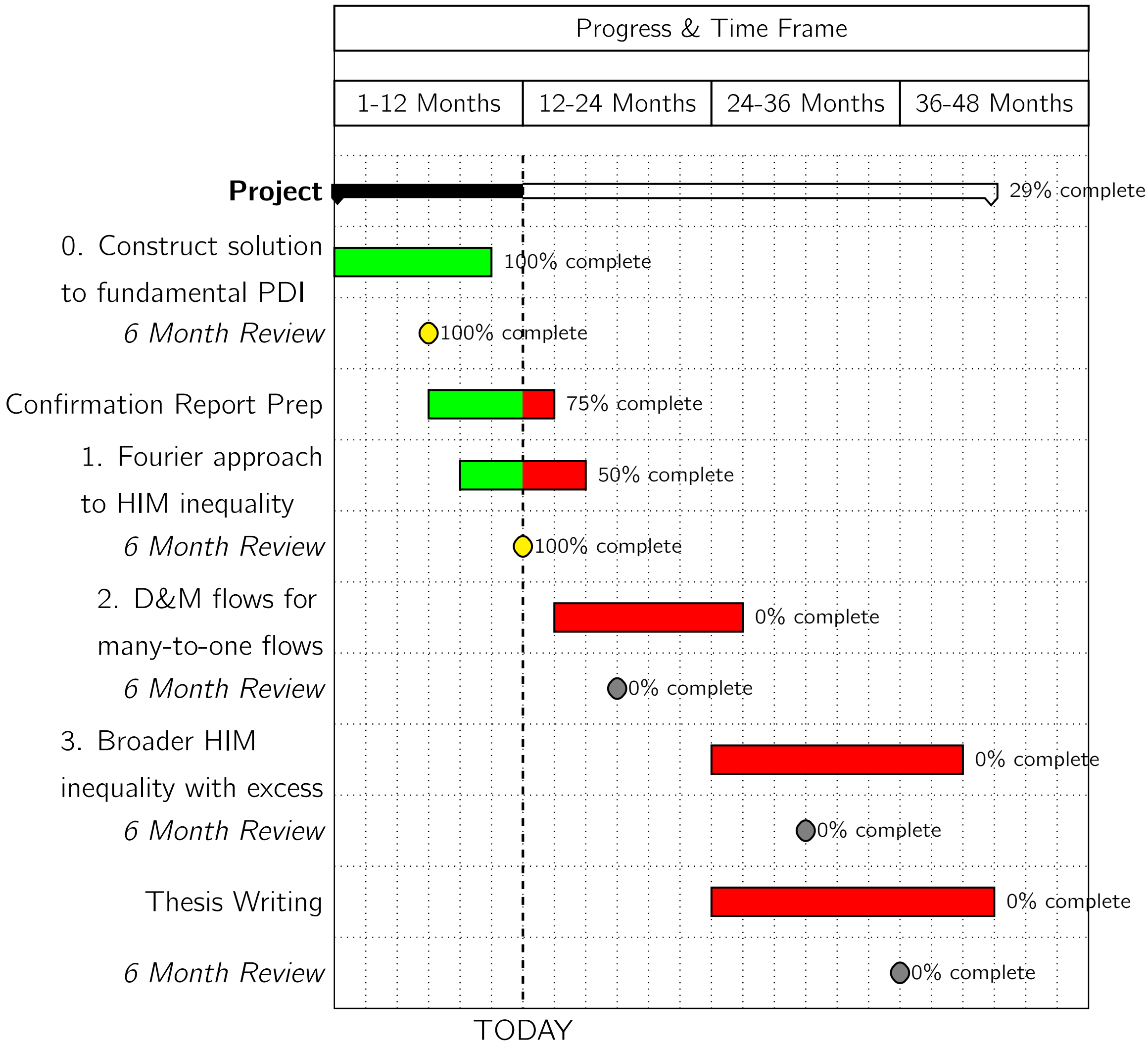
Assuming that there exists a critical value λ_{crit}, for which

$$\lambda > \lambda_{\text{crit}} \Rightarrow E_f \ngeqslant 0,$$
$$\lambda < \lambda_{\text{crit}} \Rightarrow E_f \geqslant 0,$$

we have now established a bounded interval $(\lambda_{\text{suf}}, \lambda_{\text{nec}})$ in which λ_{crit} must lie. It is likely that the value for λ_{nec} can be further decreased using just the methods described in this report with some optimisations. An exploration for methods of tiling as much a ball with a minimal amount of squares under certain conditions (we require some amount of asymmetry to avoid $I_0 = 0$) may help shed some light on how we can get the most out of the methods for bounding λ_{nec}. It is currently unclear on how to improve the bounds for λ_{suf}. We shall finish with some open questions:

- Can these results be extended to general $f \in W^{1,\infty}(B; \mathbb{R})$, with characterisation given by the value of $\|f\|_\infty$ or $\|\nabla f\|_\infty$?
- Can we find a broad class of solutions to 1.1, in the case of a convex domain, using the integral forms described in 2.4.?
- Are there ways of generating solutions to 1.1 with less symmetry, without oscillations of arbitrarily large frequency occurring in the interior of the domain?

Gantt Chart



Progress & Time Frame
1-12 Months
12-24 Months
24-36 Months
36-48 Months
Project
29% complete
0. Construct solution to fundamental PDI
100% complete
6 Month Review
100% complete
Confirmation Report Prep
75% complete
1. Fourier approach to HIM inequality
50% complete
6 Month Review
100% complete
2. D&M flows for many-to-one flows
0% complete
6 Month Review
0% complete
3. Broader HIM inequality with excess
0% complete
6 Month Review
0% complete
Thesis Writing
0% complete
6 Month Review
0% complete
TODAY

Appendices

$\mathcal{A}$ Properties of Vectors & Matrices

Proposition A.1. *If* $a_i, b_i \in \mathbb{R}^n$ *for* $i = 1, \ldots, n$, *then*

$$\begin{pmatrix} a_1 & \ldots & a_n \end{pmatrix} \begin{pmatrix} b_1 & \ldots & b_n \end{pmatrix}^{\mathsf{T}} = \sum_{i=1}^{n} a_i \otimes b_i.$$

Proposition A.2. *If* $A, B \in \mathbb{R}^{n \times n}$, *then*

$$\begin{aligned} \operatorname{cof}(A + B) &= \operatorname{cof}(A) + \operatorname{cof}(B), \\ \det(A + B) &= \det(A) + \det(B) + \langle \operatorname{cof}(A), B \rangle_{\mathsf{F}} . \end{aligned}$$

Proposition A.3. *If* $a, b, c, d \in \mathbb{R}^n$, *then*

$$\langle a \otimes b, c \otimes d \rangle_{\mathsf{F}} = \langle a, c \rangle_2 \langle b, d \rangle_2 .$$

Corollary A.4. *If* $a, b \in \mathbb{R}^n$, *then*

$$|a \otimes b|_{\mathsf{F}} = |a \otimes b|_2 = |a|_2 \cdot |b|_2 .$$

Proposition A.5. *If* $a, b \in \mathbb{R}^2$ *such that* $b^{\mathsf{T}} J a \neq 0$, *then*

$$\begin{pmatrix} a & b \end{pmatrix}^{-1} = \frac{1}{b^{\mathsf{T}} J a} \begin{pmatrix} b^{\mathsf{T}} J \\ a^{\mathsf{T}} J^{\mathsf{T}} \end{pmatrix} .$$

Proposition A.6. *If* $A \in \mathbb{R}^{2 \times 2}$, *then*

$$\begin{aligned} \operatorname{cof}(A) &= J^{\mathsf{T}} A J, \\ \det(A) &= \frac{1}{2} \langle cof(A), A \rangle_{\mathsf{F}} . \end{aligned}$$

Corollary A.7. *If* $a, b \in \mathbb{R}^2$, *then*

$$\begin{aligned} \operatorname{cof}(a \otimes b) &= Ja \otimes Jb, \\ \det(a \otimes b) &= 0. \end{aligned}$$

References

[1] J. M. Ball and R. D. James, *Fine phase mixtures as minimizers of energy*, Archive for Rational Mechanics and Analysis **100** (1987), no. 1, 13–52, doi.

[2] J.M Ball and F Murat, *W1,p-quasiconvexity and variational problems for multiple integrals*, Journal of Functional Analysis **58** (1984), no. 3, 225–253, doi.

[3] John M. Ball, *Convexity conditions and existence theorems in nonlinear elasticity*, Archive for Rational Mechanics and Analysis **63** (1976), no. 4, 337–403, doi.

[4] ______, *Some Open Problems in Elasticity*, Geometry, Mechanics, and Dynamics (Paul Newton, Philip Holmes, and Alan Weinstein, eds.), Springer-Verlag, New York, 2002, pp. 3–59, doi.

[5] J. Bevan, M. Kruzik, and J. Valdman, *A mean Hadamard inequality*, Preprint (2022).

[6] Jonathan Bevan, *On double-covering stationary points of a constrained Dirichlet energy*, Annales de l'Institut Henri Poincaré C, Analyse non linéaire **31** (2014), no. 2, 391–411, doi.

[7] Robert B. Burckel, *The Riemann Mapping Theorem*, pp. 293–343, Birkhäuser Basel, Basel, 1979, doi.

[8] Arrigo Cellina and Stefania Perrotta, *On a problem of potential wells*, J. Convex Anal **2** (1995), no. 1-2, 103–115.

[9] B Dacorogna, *Quasiconvexity and relaxation of nonconvex problems in the calculus of variations*, Journal of Functional Analysis **46** (1982), no. 1, 102–118, doi.

[10] Bernard Dacorogna, *Direct methods in the calculus of variations*, Applied Mathematical Sciences, no. 78, Springer, Berlin Heidelberg, 1989.

[11] Bernard Dacorogna and Paolo Marcellini, *Implicit partial differential equations*, Progress in Nonlinear Partial Differential Equations and Their Applications, no. v. 37, Birkhäuser, Boston, 1999.

[12] Lawrence C. Evans, *Partial differential equations*, 2nd ed ed., Graduate Studies in Mathematics, no. v. 19, American Mathematical Society, Providence, R.I, 2010.

[13] Lawrence C. Evans and Ronald F. Gariepy, *Measure theory and fine properties of functions*, Studies in Advanced Mathematics, CRC Press, Boca Raton, 1992.

[14] Paolo Marcellini, *Approximation of quasiconvex functions, and lower semicontinuity of multiple integrals*, Manuscripta Mathematica **51** (1985), no. 1-3, 1–28, doi.

[15] Charles B. Morrey, *Quasi-convexity and the lower semicontinuity of multiple integrals*, Pacific Journal of Mathematics **2** (1952), no. 1, 25–53, doi.

[16] S. Müller and V. Šverák, *Convex integration with constraints and applications to phase transitions and partial differential equations*, Journal of the European Mathematical Society **1** (1999), no. 4, 393–422, doi.

[17] S. Müller and M.A. Sychev, *Optimal Existence Theorems for Nonhomogeneous Differential Inclusions*, Journal of Functional Analysis **181** (2001), no. 2, 447–475, doi.

[18] Stefan Müller, *Variational models for microstructure and phase transitions*, vol. 1713, pp. 85–210, Springer Berlin Heidelberg, Berlin, Heidelberg, 1999, doi.

[19] Vladimír Šverák, *On the Problem of Two Wells*, Microstructure and Phase Transition (Avner Friedman, Willard Miller, David Kinderlehrer, Richard James, Mitchell Luskin, and Jerry L. Ericksen, eds.), vol. 54, Springer New York, New York, NY, 1993, pp. 183–189, doi.

[20] William P. Ziemer, *Weakly differentiable functions: Sobolev spaces and functions of bounded variation*, Graduate Texts in Mathematics, no. 120, Springer-Verlag, New York, 1989.