Calculus in Infinite Dimensions

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Finite vs Infinite Dimensional Space

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- Lebesgue & Sobolev spaces

Finite Dimensional Calculus

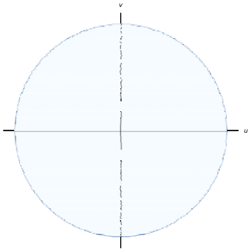
Consider a **function** $f : B \to \mathbb{R}^3$, with B the unit ball in 2D, given by

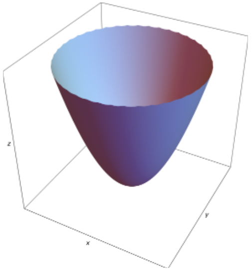
$$f(u,v) = \begin{pmatrix} u+v \\ u-v \\ u^2+v^2 \end{pmatrix}$$

Then the **gradient** is given by

$$\nabla f(u,v) = \begin{pmatrix} +1 & +1 \\ +1 & -1 \\ 2u & 2v \end{pmatrix}$$

We can use this to investigate stationary points, minima, maxima etc.





Infinite Dimensional Calculus

Now consider a **functional** $f : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$, given by

$$f(x) = x_1$$

Then the **gradient** is a sequence $\nabla f(x)$ with entries given by

$$(\nabla f(x))_i = \delta_{1,i} = \begin{cases} 1 & i = 1 \\ 0 & i > 1 \end{cases}$$

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We shall mainly focus on functionals defined on spaces of functions.

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$$\nabla f(x) = 0$$

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Taylor expand f:

$$f(x_0 + \epsilon v) - f(x_0) = \epsilon \nabla f(x_0)[v] + \epsilon^2 \nabla^2 f(x_0)[v, v] + o(\epsilon^2) \geq 0$$

for any vector v.

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for any vector v. Then

$$\nabla f(x_0) = 0$$
$$\nabla^2 f(x_0) \geq 0$$

is sufficient. However, only $\nabla^2 f(x_0) \geq 0$ is necessary.

Local Minima in Infinite Dimensions

Consider a functional

$$F[x] = \int_0^1 f(t, x(t), x'(t)) \, \mathrm{d}t \qquad x \in C^2((0,1), \mathbb{R}^d)$$

For now, assume $f \in C^1$.

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$$F(x_0 + \epsilon\varphi) - F(x_0) = \int_0^1 f(t, x_0 + \epsilon\varphi, x_0' + \epsilon\varphi') - f(t, x_0, x_0')\,\mathrm{d}t$$

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We want this to be positive for all $\varphi \in C_c^\infty(0,1)$ and small ϵ.

Local Minima in Infinite Dimensions

If we want

$$\int_0^1 \left(f_x(t, x_0, x_0') - f_{x'}(t, x_0, x_0')'\right) \varphi \,\mathrm{d}t = 0 \qquad \forall \varphi \in C_c^\infty(0,1)$$

we must have

$$\frac{\delta F}{\delta x} := f_x(t, x_0, x_0') - f_{x'}(t, x_0, x_0')' = 0 \qquad \forall t \in [0,1]$$

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$$\frac{\delta F}{\delta x} := f_x(t, x_0, x_0') - f_{x'}(t, x_0, x_0')' = 0 \qquad \forall t \in [0, 1]$$

This is known as the **Euler-Lagrange** equation. It is a second order differential equation to be solved for $x_0(t)$, provided we specify initial/boundary conditions.

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$$= \left(\mathbf{r}'(t) \otimes \mathbf{r}'(t) - \left|\mathbf{r}'(t)\right|_2^2 I\right) \frac{\mathbf{r}''(t)}{|\mathbf{r}'(t)|_2^3}$$

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When $d > 1$

$$\mathbf{r}'(t) \otimes \mathbf{r}'(t) - \left|\mathbf{r}'(t)\right|_2^2 I \neq 0$$

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This gives

$$\mathbf{r}(t) = (\mathbf{b} - \mathbf{a})t + \mathbf{a}$$

Projectiles

The action of a particle of mass m travelling along a path $\mathbf{r}(t)$, under the influence of gravity, is

$$S[\mathbf{r}] = \int_0^1 \frac{m}{2} \left|\mathbf{r}'(t)\right|_2^2 + m\mathbf{g} \cdot \mathbf{r}(t)\,\mathrm{d}t \qquad \mathbf{g} = (0, -g)$$

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If we launch a particle from a cannon at the origin and it lands at $(1, 0)$. The trajectory must solve

$$\mathbf{r}''(t) = \mathbf{g} \qquad \mathbf{r}(0) = (0, 0), \quad \mathbf{r}(1) = (1, 0)$$

Brachistochrone

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Thus we need to solve

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The solution to this is an example of a **cycloid** and can be written as

$$x(\omega\theta) = \alpha(\theta - \sin(\theta))$$
$$y(\omega\theta) = \alpha(1 - \cos(\theta))$$

for an appropriate choice of $\alpha > 0$ and ω.

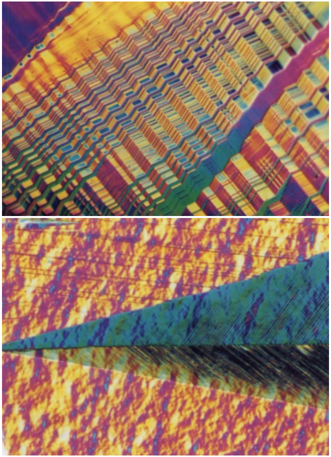
Elasticity & SME

Deformations of materials Ω are often investigated by studying functionals:

$$E[u] = \int_\Omega \varphi(\nabla u, \theta)\,\mathrm{d}V$$

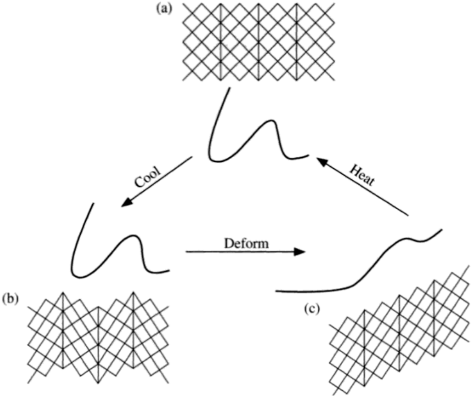
where

- E is the total energy of the material.
- φ is the stored energy density.
- u is an admissible deformation.
- θ is the temperature.



Shape Memory Effect

Changing the temperature changes the functional and thus changes the minimisers (what we observe).



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Dirichlet Energy Functional

The Dirichlet energy functional $\mathbb{D}(u)$ can be used to describe the strain energy density under a deformation u.

$$\mathbb{D}(u) = \int_B |\nabla u|_F^2 \, \mathrm{d}x \qquad u \in u_0 + W_0^{1,2}(B; \mathbb{R}^2)$$

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where, in particular, we have

$$E(u) = E_f(u) := \int_B |\nabla u|_{\mathsf{F}}^2 + f(x) \det \nabla u \, \mathrm{d}x$$

with $f(u) = 3 \log |x|_2$ (often called a 'pressure' function) [1].

A 1D Family of Polyconvex Functions

The functional

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We observe that:

- $E_\lambda(0) = 0$.
- E_λ is either unbounded below or bounded below by zero.

Hence E_λ has a global minimum iff $E_\lambda \geq 0$.

Pointwise Hadamard

We can use of the following theorem

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For any matrix $A \in \mathbb{R}^{2\times 2}$, we have $|A|_F^2 \geq 2\,|\det A|$.

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to show

$$E_\lambda(u) \geq \int_B |\nabla u(x)|_{\mathsf{F}}^2 - \frac{\lambda}{2}\,|x|_2\,|\nabla u(x)|_{\mathsf{F}}^2 \,\mathrm{d}x$$
$$= \int_B \left(1 - \frac{\lambda}{2}\,|x|_2\right) |\nabla u(x)|_{\mathsf{F}}^2 \,\mathrm{d}x \geq 0$$

for $\lambda \leq 2$.

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for $\lambda \leq 2$. Can we do better?

Fourier Series & Poincaré Inequality

Consider a Fourier expansion of u as

$$u = \sum_{j \geq 0} u^{(j)} = \frac{1}{2}\mathbf{A}_0(r) + \sum_{j>0} \mathbf{A}_j(r)\cos(j\theta) + \mathbf{B}_j(r)\sin(j\theta)$$

with $\mathbf{A}_j(1) = \mathbf{B}_j(1) = 0$ for each $j \geq 0$ [1].

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with $\mathbf{A}_j(1) = \mathbf{B}_j(1) = 0$ for each $j \geq 0$ [1].

Then we make use of a Poincaré inequality for the Fourier modes:

Poincaré Inequality for Fourier Modes

Let $j \geq 1$. Then

$$\int_B \left| u^{(j)} \right|_2^2 \mathrm{d}x \leq \frac{1}{j_0^2} \int_B \left| u_{,r}^{(j)} \right|_2^2 \mathrm{d}x$$

where j_0 is the first zero of a Bessel function.

Applying the Inequality

Applying this inequality, we find

$$E_\lambda(u) = \sum_{j\geq 0} \int_B \left|u^{(j)}_{,r}\right|_2^2 + \left|u^{(j)}_{,\tau}\right|_2^2 + \frac{\lambda}{2}\left\langle u^{(j)}, Ju^{(j)}_{,\tau}\right\rangle_2 \,\mathrm{d}x$$

$$\geq \sum_{j\geq 0} \int_B j_0^2 \left|u^{(j)}\right|_2^2 + \left|u^{(j)}_{,\tau}\right|_2^2 - \frac{\lambda}{2}\left|u^{(j)}\right|_2 \left|u^{(j)}_{,\tau}\right|_2 \,\mathrm{d}x$$

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Applying this inequality, we find

$$
\begin{aligned}
E_\lambda(u) &= \sum_{j\geq 0} \int_B \left|u_{,r}^{(j)}\right|_2^2 + \left|u_{,\tau}^{(j)}\right|_2^2 + \frac{\lambda}{2}\left\langle u^{(j)}, J u_{,\tau}^{(j)}\right\rangle_2 \,\mathrm{d}x \\
&\geq \sum_{j\geq 0} \int_B j_0^2 \left|u^{(j)}\right|_2^2 + \left|u_{,\tau}^{(j)}\right|_2^2 - \frac{\lambda}{2}\left|u^{(j)}\right|_2 \left|u_{,\tau}^{(j)}\right|_2 \,\mathrm{d}x \\
&= \sum_{j\geq 0} \int_B \mathbf{U}_j^\mathsf{T} M_\lambda \mathbf{U}_j \,\mathrm{d}x \qquad \mathbf{U}_j = \begin{pmatrix} \left|u^{(j)}\right|_2 \\ \left|u_{,\tau}^{(j)}\right|_2 \end{pmatrix} \quad M_\lambda = \begin{pmatrix} j_0^2 & -\frac{\lambda}{4} \\ -\frac{\lambda}{4} & 1 \end{pmatrix}
\end{aligned}
$$

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$$E_\lambda(u) = \sum_{j\geq 0} \int_B \left|u_{,r}^{(j)}\right|_2^2 + \left|u_{,\tau}^{(j)}\right|_2^2 + \frac{\lambda}{2}\left\langle u^{(j)}, Ju_{,\tau}^{(j)}\right\rangle_2 \,\mathrm{d}x$$

$$\geq \sum_{j\geq 0} \int_B j_0^2 \left|u^{(j)}\right|_2^2 + \left|u_{,\tau}^{(j)}\right|_2^2 - \frac{\lambda}{2}\left|u^{(j)}\right|_2 \left|u_{,\tau}^{(j)}\right|_2 \,\mathrm{d}x$$

$$= \sum_{j\geq 0} \int_B \mathbf{U}_j^\mathsf{T} M_\lambda \mathbf{U}_j \,\mathrm{d}x \qquad \mathbf{U}_j = \begin{pmatrix} \left|u^{(j)}\right|_2 \\ \left|u_{,\tau}^{(j)}\right|_2 \end{pmatrix} \quad M_\lambda = \begin{pmatrix} j_0^2 & -\frac{\lambda}{4} \\ -\frac{\lambda}{4} & 1 \end{pmatrix}$$

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Then we have that $\lambda \leq 4 j_0 \Rightarrow M_\lambda \geq 0 \Rightarrow E_\lambda \geq 0$.

Here $4j_0 \approx 9.619$ is a significant improvement on the previous bound of $\lambda \leq 2$.

The Approach

We shall first construct a solution $u^* \in W_0^{1,2}$ to

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A necessary condition can then be derived:

$$\lambda \leq \frac{2(\pi - |K|)}{|\mathcal{I}_0|}$$

Partial Differential Inclusions

The PDI

$$\nabla u \in \mathrm{O}(2)$$

is an example of the **probelm of potential wells**:

The Problem of Potential Wells [2, 4, 5]

$$\nabla u \in \bigcup_i \mathrm{SO}(d) A_i \qquad A_i \in \mathbb{R}^{d \times d}$$

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The following is known for this PDI:

Theorem (Dacorogna and Marcellini) [3]

There exists a dense set of solutions in $W^{1,\infty}(Q; \mathbb{R}^2)$.

Theorem (Liouville) [4]

All solutions are piecewise affine.

Constructed Solution

A solution has been constructed for the 3D equivalent of this PDI [2].

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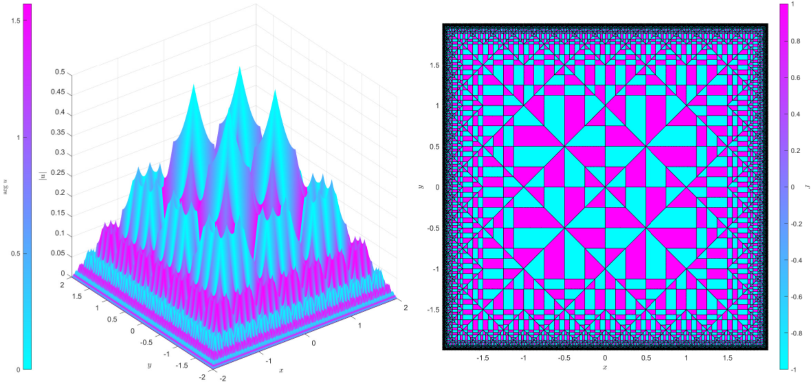

Figure: Constructed solution u^* and its Jacobian $\det \nabla u^*$

Tiling the Solution

We can then tile the ball with this solution.

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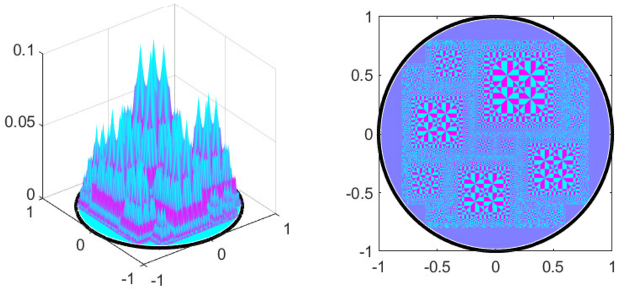

Figure: Tiled Solution with 15 Squares

Numerical Results

We will make use of a simple tiling of 2 squares with

$$c_1 = \begin{pmatrix} +0.1 \\ +0.3 \end{pmatrix} \quad w_1 = 1.0 \qquad c_2 = \begin{pmatrix} -0.3 \\ -0.5 \end{pmatrix} \quad w_1 = 0.6$$

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This leads to a bound of $\lambda \leq 148.446$.

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- $\lambda \leq 9.619$ is sufficient for $E_\lambda \geq 0$.
- $\lambda \leq 148.446$ is necessary for $E_\lambda \geq 0$.

With further optimisations, we could try to refine these bounds.

We could try to find λ_{crit} such that $E_\lambda \geq 0$ iff $\lambda \leq \lambda_{\text{crit}}$.

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